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THE ALGORITHMIC THEORY OF INFORMATION AND ITS APPLICATIONS TO GEOMETRIC MEASURE THEORY

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Abstract

Classical notions of fractal dimension measure the size of sets based on their covering properties. In a similar vein, algorithmic information theory defines the effective dimension of a point in terms of its ability to be covered to arbitrary precision following an effective procedure. The Point-to-Set Principles of J. Lutz and N. Lutz formalize the connection between certain classical fractal dimensions and effective dimensions, enabling the use of algorithmic techniques to prove results in geometric measure theory. While these connections have led to fruitful developments, the full relationship between algorithmic information theory and geometric measure theory remains incompletely understood, and an active dialogue has emerged between the two communities.

This dissertation contributes to that dialogue by deepening the theoretical connections in several directions.

We begin by reviewing the various approaches to algorithmic information theory via randomness tests, incompressibility, and mass distribution. We highlight the relevant relationships between these approaches, and discuss how they characterize certain fractal dimension notions in a pointwise manner.

We move on to demonstrate the robustness of certain complexity and effective dimension notions over Euclidean space. Many variants of Kolmogorov complexity satisfy a symmetry of information, or chain rule. In symbols, the chain rule takes the form $K(a, b) \approx K(a) + K(b \mid a)$, where a and b are considered to be finite binary strings, and approximate equality holds up to sub-linear terms in their lengths. Here, $K(b \mid a)$ denotes the conditional complexity of b given a. The chain rule plays an essential role in most applications of algorithmic information theory, and, to some extent, has been extended to Euclidean space. We prove a general chain rule for the usual lift of conditional, prefix-free Kolmogorov complexity to Euclidean space. And we discuss multiple senses in which this complexity notion is robust, including that conditional effective Hausdorff dimension is invariant under bi-computable, bi-Lipschitz continuous transformations.

Next, we identify for real maps some weaker continuity conditions under which both prefix-free Kolmogorov complexity and effective dimension behave well. This involves extending the results of N. Lutz and D. Stull bounding from the below the effective dimension of points along the graphs of planar lines. We introduce a function family which we term a *computable absolutely Lipshcitz family (CALF)* which admits a similar result for the effective dimension of points along their graphs. In fact, the effective dimension result is viewed as the limit of an analogous statement regarding the Kolmogorov complexity on finitary inputs. We deduce a result about the dimension spectrum for the graphs of functions from a CALF.

Later, we extend foundational results in algorithmic information theory to a broad class of metric spaces that admit nets in the sense of D. Larman, C. Rogers, and R. Davies. Net spaces were partially introduced in order to extend beyond Euclidean space a result by A. Besicovitch related to finding inside any compact set of infinite measure a compact subset of non-zero, finite measure. Besicovitch's proof made essential use of the countable collection of dyadic cubes over Euclidean space, but the method is workable over more general nets. We extend algorithmic information theory to net spaces by recreating several optimality results for semimeasures and outer measures, extending complexity notions, confirming various effective dimension coincidences, and deducing point-to-set principles for both Hausdorff dimension and the Hausdorff outer measures. We also discuss examples of net spaces, such as compact metric spaces and Polish spaces, and their effectivizations.

Finally, we apply these tools to prove both new and existing results in classical geometric measure theory. We use Kolmogorov complexity to prove two results essential to T. Orponen's combinatorial proof of the Marstrand-Mattila Projection Theorem under the assumption that Hausdorff and packing dimensions agree. Incompressibility arguments also prove a simple bound on Hausdorff dimension under locally-Lipschitz transformations. We apply this bound over several geometries, producing some modest bounds on the Hausdorff dimension of both orthogonal and radial projections. We also obtain a density result over net spaces under some modest assumptions, which implies an analog of Besicovitch's result on the existence of compact subsets of non-zero, finite measure.

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Chapter 1
Introduction

1.1 Introduction

This dissertation concerns the application of algorithmic information theory (AIT) to geometric measure theory (GMT). This investigation builds on the mounting evidence that mathematics under algorithmic restrictions offers insights into–and refinements to–the study of the fractal properties of sets.

The origins of algorithmic information theory began in the 1960s with the formalization of *Solomonoff induction*: a rule for selecting an hypothesis for the source from which some given data was produced. The rule substitutes a non-computable, universal prior distribution into Bayes' rule, and rewards algorithmically-simpler explanations [61]. R. Solomonoff's conception of information content is most closely related to the algorithmic probability of producing a piece of data with respect to a universal Turing machine.

An independent origin came just a few years later when A. Kolmogorov defined the algorithmic (plain) complexity of individual objects, thereby refining the concept of classical Shannon information which only describes the mean information content across a distribution [25, 26]. So, Kolmogorov's concept of information content was based on the degree of incompressibility with respect to a universal Turing machine.

And shortly after, G. Chaitin also independently introduced plain complexity and established its symmetry of information property, as well as establishing some initial notions of algorithmic randomness for infinite binary strings based on attaining maximal complexity along their prefixes [7].

P. Martin-Löf characterized algorithmic randomness for an infinite binary sequence instead as the inability to cover it to arbitrary precision by effectively open statistical tests [42]. This adjusted A. Church's original attempt: computable randomness, meant to achieve the intuitive property due to von Mises that random sequences ought to pass all standard statistical tests such as the Law of Large Numbers.

C. Schnorr showed that both Chaitin's and Martin-Löf's notions of algorithmic randomness for infinite binary sequences were equivalent to yet another: those sequences on which any effective betting strategy (i.e., *constructive martingale*) would fail to generate arbitrary gains when betting on subsequent bits [58]. Thus, what is now commonly known as *Martin-Löf randomness* is considered robust for its equivalent characterizations in terms of Chaitin's notion of incompressibility, Martin-Löf's notion of typicality, and Schnorr's notion of unpredictability.

Moreover, L. Levin confirmed the equivalence between Solomonoff, Kolmogorov, and Chaitin's approaches to quantifying algorithmic information content [28]. Levin's Coding Theorem equated Solomonoff's universal prior with $2^{-K(\cdot)}$, where $K(\cdot)$ denotes *prefix-free* or *self-delimiting Kolmogorov complexity*, or the minimal description-length of some data with respect to a universal, prefix-free Turing machine.

Whether viewed as the branch of theoretical computer science dedicated to the properties of computably-generated objects, or as a refinement of classical Shannon information theory by measuring the information content of individual objects, or as a project to formalize randomness: algorithmic information theory clearly offers some extra structure to otherwise "classical" mathematics, or the mathematics unconstrained by effectivity.

Just as any natural, geometric object has an assignable geometric dimension that corresponds to the right "exponential factor" by which to measure its volume, the goal of many fractal dimension notions is to assign to a set an exponent appropriate to its geometry [10]. For instance, the *Hausdorff dimension* of a set is roughly defined as the critical value of s for which the volume of the set as measured using exponent s switches from being infinite to null. Thus, Hausdorff dimension is naturally associated to the family of *s-dimensional Hausdorff outer measures* $\mathcal{H}^s$ (defined in Section 1.3). These outer measures refine Lebesgue measure on null sets based on their covering properties. Yet, a further refinement to $\mathcal{H}^s$-nullness is exhibited by *partial randomness*.

Generalizing the constructive martingales of Schnorr, J. Lutz introduced *constructive s-supergales*, where $s \geq 0$ is a dimension parameter [31]. Whenever a constructive s-supergale could make arbitrary gains betting on the subsequent bits of a given sequence (i.e., *succeed* on the sequence), then that sequence would not be considered random with respect to s. Then, Lutz defined the *constructive dimension* of a sequence to roughly be the minimal s on which some constructive s-supergale could succeed.

Shortly after, E. Mayordomo elucidated the relationship between Lutz's constructive dimension and Kolmogorov complexity. In particular, the constructive dimension of a sequence matches L. Staiger's limit inferior of the normalized Kolmogorov complexities of the sequence's prefixes [45, 63].

Both Lutz's and Mayordomo's contributions were built on decades of prior work toward understanding Kolmogorov complexity, including bounds on plain complexity by Shannon entropy [62], as well as work by both B. Ryabko [56, 57] and Staiger [64] partially relating the limit inferior of the normalized Kolmogorov complexity of prefixes to the speed at which a computable martingale could make arbitrary gains on a given sequence.

Generalizing the work of Schnorr and Chaitin, K. Tadaki introduced notions of Martin-Löf tests and partial randomness tailored to a dimension parameter $s \geq 0$ [68], which in turn provided yet another characterization of constructive dimension as roughly the minimal value of s for which a set could be covered by one of these tests. Thus, this effective notion of dimension serves as a refinement to Martin-Löf randomness, and similarly may be characterized in terms of incompressibility, typicality, and unpredictability.

In effect, effective dimension quantifies just *how random* a sequence behaves in the limit. In particular, it is possible for singleton sets to have nonzero effective dimension (in fact, most have full effective dimension). These would correspond to points which are not covered by a Martin-Löf-test appropriate for some dimension $s > 0$; or, equivalently, those admitting highly incompressible prefixes; or, those on which any constructive s-supergale would not succeed at making arbitrary gains for some $s > 0$.

In the recent past, computability theorists and geometric measure theorists alike have used some notions of effective dimension to prove classical results about the fractal dimension of certain sets. The so-called *Point-to-Set Principles* by J. Lutz and N. Lutz have almost always served as the bridge between effective and "classical" results about Hausdorff or packing dimensions in Euclidean space [34]. For example, their formula in Theorem 1.12.1 characterizes the Hausdorff dimension of a set in a pointwise manner: a strange result given that all singleton sets are classically zero-dimensional. But, as we have discussed, most points are not forthright in revealing their own smallness by effective means. Essentially, sets of large fractal dimension must contain many sufficiently random points.

There are still many open questions in regards to the potential applications of algorithmic information theory to geometric measure theory, as well as to what settings AIT applies.

The remainder of this dissertation is organized as follows:

- **Chapter 1**: We fix some notation and review the basic definitions and results relevant to our discussion about AIT and GMT. Central to our work are the outer measures induced by premeasures on a space, various cover and test notions from algorithmic randomness, and measures of algorithmic information content such as Kolmogorov complexity and semimeasures. In particular, Theorem 1.8.3 relates three notions from the partial randomness literature: success by a lower-semicomputable continuous semimeasure, success by a constructive supergale, and covering by a sequence of strong Solovay tests with rapidly shrinking weights.

- **Chapter 2**: We establish some basic properties of the lift of Kolmogorov complexity to Euclidean space. Theorem 2.1.1 shows that the order of optimizers in the definition of conditional prefix complexity makes no difference up to sub-linear factors. Theorem 2.1.11 proves a conditional symmetry of information for conditional prefix complexity. Lemma 2.2.2 confirms a modulus processing inequality for conditional prefix complexity under computable, uniformly continuous maps, which implies conditional dimension is invariant under bi-computable, bi-Lipschitz continuous maps in Theorem 2.2.4.

- **Chapter 3**: We generalize some results characterizing the distribution of effective dimension in the Euclidean plane. Our method works for any family of curves with strong, uniform-continuity properties, which we call a *computable absolutely Lipschitz family* (CALF). Theorem 3.3.3 proves a complexity bound that holds for the dyadic rational reals, leading to an effective dimension result for reals in Theorem 3.4.2. By viewing the collection of non-vertical, planar lines as given by a CALF, the lower bound result for points on lines by N. Lutz and D. Stull follows. In Theorem 3.5.3, we show the Hausdorff dimension spectrum for a CALF satisfies a straightforward bound. The work in this section arose under the supervision of Linda Westrick.

- **Chapter 4**: We generalize AIT to metric spaces admitting collections of subsets called nets, and effectivize some classical constructions on such spaces. Propositions 4.3.4 and 4.3.9 prove the existence of optimal lower-semicomputable mesh semimeasures in both the continuous and discrete cases. Theorem 4.4.16 proves N. Lutz's outer measure $\boldsymbol{\kappa}$ is locally optimal over any layered-disjoint net. Theorem 4.5.16 summarizes the asymptotic coincidences we establish between various effective dimension notions over a net space. Theorems 4.6.2, 4.6.7, and 4.6.9, as well as Corollary 4.6.3, confirm that the point-to-set principles for Hausdorff dimension and the family of s-dimensional Hausdorff outer measures hold over net spaces which are rich with net measures. We also construct nets on effective versions of compact metric spaces (Corollary 4.7.6) and Polish spaces (Proposition 4.7.13). The work in this section arose under the supervision of Jan Reimann.

- **Chapter 5**: We use Kolmogorov complexity to prove two geometric measure theoretic results essential to T. Orponen's combinatorial proof of the Marstrand-Mattila Projection Theorem. These are Propositions 5.1.5 and 5.1.6, which arose from joint work with Ryan Bushling under the supervision of Jan Reimann. Next, we give a Kolmogorov complexity-based proof for a simple bound about Hausdorff dimension under locally-Lipschitz transformations: Theorem 5.2.4. This result applies to various geometries, producing some modest bounds on the Hausdorff dimension of orthogonal and radial projections in Propositions 5.2.5 and 5.2.6, respectively. Finally, we show that a new density result: Theorem 5.3.4, holds over a wide class of net spaces. This implies an analog of Besicovitch's result on the existence of compact subsets of non-zero, finite measure to such net spaces. These last two results arose from joint work with Emma Gruner under the supervision of Jan Reimann.

- **Chapter 6**: We offer some concluding remarks and suggest opportunities for further investigation.

1.2 Notation

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Q}$ the set of rational numbers, and ω the set of natural numbers (including 0). Let $\mathbb{D} \subset \mathbb{Q}$ denote the set of all dyadic rational numbers (i.e., numbers of the form $k/2^l$ for integers $k, l \geq 0$), and $\mathbb{D}_r$ the set of all r-dyadic rational numbers (i.e., all simplified dyadic rationals where the power of 2 in the denominator is no more than r). Let 2^n denote the set of length-n binary strings, $2^{\leq n}$ the set of all binary strings length no greater than n, $2^{<\omega}$ the set of finite binary strings of any length, and analogous definitions for each of $2^{\geq n}$, $2^{>n}$, 2^ω, and $2^{\leq \omega}$. Similarly, $\omega^{<\omega}$ will denote the set of finite sequences of natural numbers, and ω^ω the set of infinite sequences of natural numbers. Generally, for any space X, interpret X^m as the set of length-m tuples over X. In particular, elements of $(2^{\leq n})^m$ are m-tuples of binary strings of a common length less than or equal to n.

If $\delta > 0$ and A is a subset of a metric space (Ω, d), then let us define the *δ-neighborhood of A* to be $B_\delta(A) := \{x \in \Omega : d(x, A) < \delta\}$. Then for any $x \in \Omega$, the set $B_\delta(x) := B_\delta(\{x\})$ is the open ball of radius δ about x.

The cardinality of a set A is denoted by $|A|$. We also use $|\cdot|$ for the absolute value on $\mathbb{R}$. Use λ to denote the Lebesgue measure on both 2^ω and $\mathbb{R}$. For any $p \in [0, \infty]$, $||\cdot||_p$ denotes the L^p-norm on Euclidean space, and we may simply write $||\cdot||$ for the standard L^2-norm unless specified otherwise. While $\log_b$ will denote the base-b logarithm for any $0 < b \neq 1$, we will always take log to mean the base-2 logarithm. Also let $\mathcal{Q}^m$ be the collection of all the m-dimensional dyadic cubes in $\mathbb{R}^m$, consisting of the sets $Q(z, a, r)$ of the form:

$$[z_0 + a_0 \cdot 2^{-r}, z_0 + (a_0 + 1) \cdot 2^{-r}) \times \cdots \times [z_{m-1} + a_{m-1} \cdot 2^{-r}, z_{m-1} + (a_{m-1} + 1) \cdot 2^{-r}),$$

where $z \in \mathbb{Z}^m$, $a \in \omega^m$, and $r \in \omega$. Collect into $\mathcal{Q}^m_r$ all the m-dimensional dyadic cubes $Q(z, a, r)$ with side-length 2^{-r}.

It will often happen that certain equalities hold only up to an additive or multiplicative constant. For this reason we use the following notation: a statement of the form $f(x) \leq^+ g(x)$ for all $x \in X$ means that there exists a constant $c \in \mathbb{R}$ such that $f(x) \leq g(x) + c$ for all $x \in X$. Similarly $f(x) =^+ g(x)$ for all $x \in X$ means that $f(x) \leq^+ g(x) \leq^+ f(x)$ for all $x \in X$. Similarly, $f(x) \leq^* g(x)$ for all $x \in X$ will mean there exists a constant $c \in \mathbb{R}$ such that $f(x) \leq c \cdot g(x)$ for all $x \in X$, and similar for $f(x) =^* g(x)$. Similarly, the notation $f = O_{\boldsymbol{v}}(g)$ is to be interpreted as $f \leq^* g$ everywhere, where the implicit multiplicative constant may depend on the components in $\boldsymbol{v}$. And the notation $o(r)$ is used to represent a *sub-linear term*, or a term which satisfies $\lim_{r\to\infty} \frac{o(r)}{r} = 0$.

The length of a string $\sigma \in 2^{\leq\omega}$ may be denoted by $\text{len}(\sigma)$ (infinite whenever $\sigma \in 2^\omega$). A string $\tau \in 2^{\leq\omega}$ *extends* another string $\sigma \in 2^{\leq\omega}$ if $\tau(n) = \sigma(n)$ for all $n < \text{len}(\sigma)$, and this is denoted by $\sigma \preceq \tau$. Moreover, τ is a *proper extension* of σ if $\sigma \preceq \tau$ yet $\sigma \neq \tau$. Two strings σ and τ are *comparable* (denoted $\sigma \mid\mid \tau$) if either $\sigma \preceq \tau$ or $\tau \preceq \sigma$. String concatenation is denoted by $\sigma^\frown\tau$ (or sometimes $\sigma\tau$, omitting the concatenation symbol) for any strings $\sigma \in 2^{<\omega}$ and $\tau \in 2^{\leq\omega}$. Formally, this is defined as the unique string of length $\text{len}(\sigma) + \text{len}(\tau)$ such that $\sigma \preceq \sigma^\frown\tau$ and $(\sigma^\frown\tau)(\text{len}(\sigma) + n) = \tau(n)$ for all $0 \leq n < \text{len}(\tau)$.

A collection of finite binary strings $S \subseteq 2^{<\omega}$ is called *prefix-free* if no two elements of

S are comparable, i.e., $(\forall \sigma, \tau \in S)[\neg(\sigma \mid\mid \tau)]$. Then for any $S \subseteq 2^{<\omega}$, denote by $\widehat{S}$ the collection of all minimal elements of S with respect to the prefix partial relation, i.e.,

$$\widehat{S} := \{\sigma \in S : \neg(\exists \tau \preceq \sigma)[\tau \in S]\}.$$

Notice that $\widehat{S}$ will always be a prefix-free subset of S.

For any finite binary string $\sigma \in 2^{<\omega}$, denote by $[\sigma]$ the *cylinder set above* σ comprising all infinite extensions of σ. Collect into $\mathcal{B}$ all of the cylinder sets above the finite strings from $2^{<\omega}$. Then $\mathcal{B}$ serves as a clopen basis for the product topology on 2^{ω}, in which every open set is of the form $[S] := \bigcup_{\sigma \in S}[\sigma]$, where $S \subseteq 2^{<\omega}$. Also, let $[\![\sigma]\!] := \{\tau \in 2^{<\omega} : \sigma \preceq \tau\}$ denote the collection of finite binary strings extending σ. The truncation of a binary string $\sigma \in 2^{\leq\omega}$ up to the first $r \geq 0$ places is denoted by $\sigma \upharpoonright r = (\sigma(i) : i < \min\{r, \text{len}(\sigma)\})$.

The length of a string tuple $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m)$ is defined as

$$\text{len}(\sigma) := \min\{\text{len}(\sigma_1), \text{len}(\sigma_2), ..., \text{len}(\sigma_m)\},$$

or infinite when all components are of infinite length. We may extend the prefix relation to string tuples $\sigma, \tau \in (2^{\leq\omega})^m$ component-wise,

$$\sigma \preceq \tau :\iff (\sigma_1 \preceq \tau_1) \wedge (\sigma_2 \preceq \tau_2) \wedge \cdots \wedge (\sigma_m \preceq \tau_m).$$

The truncation of a real number $x \in \mathbb{R}$ up to precision-level $r \in \omega$ may be defined as $x \upharpoonright r := 2^{-r} \cdot \lfloor x \cdot 2^r \rfloor$. Under the standard association between real numbers and their infinite binary expansions (under the convention of only permitting infinite tails of zeros), these two notions of truncation agree in the following sense. Without loss of generality, consider $x \in [0, 1)$ and let x have the infinite binary expansion $0.x(0)x(1)x(2)\cdots$. Then, for any $r \in \omega$, it holds that its length-r truncation as a string (followed by an infinite tail of zeros) $0.x(0)x(1)\cdots x(r-1)^\frown 0^\omega$ is the infinite binary expansion of the real number $x \upharpoonright r$.

Truncation for a real tuple $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m$ is taken component-wise:

$$x \upharpoonright r := (x_1 \upharpoonright r, x_2 \upharpoonright r, ..., x_m \upharpoonright r) \in B_{2^{-r}\sqrt{m}}(x) \cap \mathbb{D}_r^m.$$

Consider the collections of natural numbers: ω, finite binary strings: $2^{<\omega}$, or dyadic rationals: $\mathbb{D}$. For the purposes of computability theory, we assume that between each pair of these sets there exists a computable bijection. Other collections can be added to this list, including the cylinder sets: $\mathcal{B}$, all the finite subsets of ω, the integers: $\mathbb{Z}$, the rationals:

$\mathbb{Q}$, the collection of finite sequences of natural numbers: $\omega^{<\omega}$, and finite-products of such sets. We say that each such collection is composed of *finitary* objects, also known as *hereditarily finite objects*. Whenever a definition is appropriate for any finitary object, we will state the definition using the symbol $\mathscr{N}$ as a placeholder for any one of these collections.

For instance, given a standard collection of finitary objects $\mathscr{N}$ as described above and an element $n \in \mathscr{N}$, one might evaluate $\text{len}(\cdot)$, truncation, or the prefix relation on the finite binary string to which n is associated under a canonical, computable bijection between $\mathscr{N}$ and $2^{<\omega}$.

Finitary objects often give rise to *infinitary* objects via some limiting process and thus admit a notion of extension. For instance, any infinite binary string $x \in 2^\omega$ is the limiting sequence of its prefixes $x \upharpoonright r \in 2^{<\omega}$ for $r \in \omega$. Any real $x \in \mathbb{R}$ is definable by Dedekind cuts over the rationals $\mathbb{Q}$. Any infinite subset of ω is the countable union of finite, nested subsets of ω. Mappings between spaces of infinitary objects depend on the context, and each space of infinitary objects can take on extra metric or topological structure, so we will treat these spaces individually. Other examples include the set of countably-infinite sequences of natural numbers: ω^ω, and the set of infinite subsets of ω and their characteristic functions.

1.3 Measures and Hausdorff Dimension

1.3.1 Measures

We begin by reviewing premeasures and their associated outer measures. For more details, refer to [53] and [55].

Definition 1.3.1. Let $\mathcal{C}$ be a collection of subsets of a set Ω containing $\emptyset$. Then a *premeasure* on $\mathcal{C}$ is a function $\rho : \mathcal{C} \to [0, +\infty]$ and satisfies $\rho(\emptyset) = 0$.

In the setting of Cantor space, it is customary to consider a premeasure on 2^ω to be any non-negative mapping defined on the cylinder sets $\rho : \mathcal{B} \to [0, +\infty]$. Of course, such maps are in one-to-one correspondence with maps defined on the finite binary strings $\rho : 2^{<\omega} \to [0, +\infty]$. We may freely interchange between these two types by identifying any $\sigma \in 2^{<\omega}$ with its cylinder set $[\sigma] \in \mathcal{B}$.

In metric spaces, there is a standard, general method (introduced as "Method II" in [55]) to induce an outer measure from a given premeasure.

Theorem 1.3.2 (Theorem 15 of [55])**.** *Let ρ be a premeasure on a collection $\mathcal{C}$ of subsets of a metric space (Ω, d). Define for each $X \subseteq \Omega$ and $\delta > 0$:*

$$\mathcal{H}^\rho_\delta(X) := \inf_{(C_i)_{i\in\omega}\subseteq\mathcal{C}} \left\{\sum_i \rho(C_i) : \mathrm{diam}_d(C_i) \leq \delta, \bigcup_i C_i \supseteq X\right\}, \quad \textit{and} \quad \mathcal{H}^\rho(X) := \sup_{\delta>0} \mathcal{H}^\rho_\delta(X).$$

Then, $\mathcal{H}^\rho$ is an outer measure on Ω.

We call $\mathcal{H}^\rho$ the *Method II outer measure associated to the premeasure ρ.* And for each $\delta > 0$, we also call $\mathcal{H}^\rho_\delta$ the *ρ-dimensional δ-content* measure.

In [55], C. Rogers also defines an alternative "Method I" which differs from Method II in that it does not enforce any upper bound on the diameter of the cover elements used in the infimum:

$$\mathcal{H}^\rho_{\mathrm{I}}(X) := \inf_{(C_i)_{i\in\omega}\subseteq\mathcal{C}} \left\{\sum_i \rho(C_i) : \bigcup_i C_i \supseteq X\right\}.$$

This outer measure is also sometimes denoted $\mathcal{H}^\rho_\infty$. While this method also produces an outer measure from a premeasure ρ which agrees on null sets with $\mathcal{H}^\rho$, Rogers points out that only the results of Method II can be guaranteed to be additive on sets which are separated by positive distance [55]. We proceed with Method II as our preferred way to produce outer measures from premeasures.

Cantor space: 2^ω, is a metric space when imbued with the standard metric,

$$d(x, y) := \begin{cases} 2^{-N} & \text{if } x \neq y \text{ and } N = \min\{n \in \omega : x(n) \neq y(n)\}, \\ 0 & \text{if } x = y. \end{cases}$$

Note that this metric is compatible with the product topology generated by the cylinder sets in $\mathcal{B}$.

1.3.2 Hausdorff Measures

An important class of outer measures on any metric space (Ω, d) are the *Hausdorff outer measures*, which are those Method II outer measures induced from premeasures whose values depend only on the diameter of the set. First, declare a *dimension function* to be any non-negative, non-decreasing, continuous-on-the-right function h defined on all non-negative real numbers and satisfying $h(t) = 0 \iff t = 0$. One may associate to any dimension function h its *Hausdorff premeasure* defined on all $X \subseteq \Omega$ as follows:

$$\rho_h(X) := (h \circ \mathrm{diam}_d)(X).$$

In Cantor space, $\rho_h(\sigma) = h\left(2^{-\mathrm{len}(\sigma)}\right)$ for any $\sigma \in 2^{<\omega}$. So, any Hausdorff premeasure on 2^ω is *length-invariant*, in that $\rho_h(\sigma)$ only depends on the length of σ. Later, we will briefly focus on *convex* Hausdorff premeasures ρ over Cantor space, which satisfy:

$$\rho(\sigma) \leq \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1),$$

for all $\sigma \in 2^{<\omega}$. It is easy to see that all length-invariant, convex Hausdorff premeasures over 2^ω are also *strongly convex*, meaning for all $\sigma \in 2^{<\omega}$ and $i \in \{0, 1\}$,

$$\rho(\sigma^\frown i) \geq \frac{\rho(\sigma)}{2}.$$

Essential to fractal geometry is the family of s-dimensional Hausdorff premeasures $\rho_s := \rho_{h_s}$, where $h_s : t \mapsto t^s$ is the dimension function associated to $s \geq 0$. If $\delta > 0$, let $\mathcal{H}^s_\delta := \mathcal{H}^{\rho_s}_\delta$ denote the *s-dimensional Hausdorff δ-content*, and $\mathcal{H}^s := \mathcal{H}^{\rho_s}$ the *s-dimensional Hausdorff outer measure*.

For any outer measure μ on a metric space (Ω, d), we say a set $X \subseteq \Omega$ is *μ-null* if $\mu(X) = 0$. Any $\mathcal{H}^s$-null X is also called *s-null*.

Suppose $\mathcal{H}^s(X) < \infty$ for some $s \geq 0$. This implies that X is s_+-null for any $s_+ > s$ too. Alternatively, if $\mathcal{H}^s(X) > 0$ for some $s \geq 0$, then it must be that $\mathcal{H}^{s_-}(X) = \infty$ for all $0 \leq s_- < s$. So, the shape of the function $s \mapsto \mathcal{H}^s(X)$ is either constant with value 0 or ∞, or a step from ∞ to 0 (where, at the critical value of s where the step occurs, $\mathcal{H}^s(X)$ could take on any value in $[0, \infty]$). This motivates the following definition for the *Hausdorff dimension* of a set X [12, 16]:

$$\dim_{\mathrm{H}}(X) := \inf\{s \geq 0 : X \text{ is } s\text{-null}\}, \tag{1.1}$$

or $+\infty$ if the infimum is taken over the empty set.

1.4 Computability Theory

1.4.1 Computability by Machines

We briefly review the necessary components of computability theory for discussing algorithmic randomness. In this dissertation, we accept Turing's model of computation as the "right" way to characterize *algorithmic* or *effective* procedures. According to the Church-Turing thesis, the subsets of the natural numbers or functions from natural numbers to natural numbers which are "intuitively algorithmic" are exactly those which may be computed by Turing machines. See [11] for more on the formalism of Turing machines.

To begin, we will consider partial functions of the form $\Phi :\subseteq \omega \to \omega$. If $\mathrm{dom}(\Phi) = \omega$, call Φ *total*. Each Turing machine M *computes* a partial function $\Phi_M :\subseteq \omega \to \omega$ satisfying $n \in \mathrm{dom}(\Phi_M) \iff M(n)\downarrow$, and $n \in \mathrm{dom}(\Phi_M)$ implies $M(n)\downarrow= \Phi_M(n)$. All such Φ_M are called *partial computable (p.c.)* functions. Moreover, whenever $A = \mathrm{dom}\,\Phi_M$ is the domain of a Turing machine M, then M is said to *computably enumerate* A, making A *computably enumerable (c.e.)*. If both A and its complement $\omega \setminus A$ are c.e., then A is called *computable*.

Proposition 2.2.2 of [11] helps to motivate the name "computably enumerable." It states that A is c.e. if and only if either A is empty or there is a total computable function Φ from ω onto A. That is, the non-empty c.e. sets are exactly those enumerated in some order by a Turing machine.

By the Enumeration Theorem (see Theorem 2.1.2 of [11]), there exists a single algorithm to enumerate all partial computable functions $(\Phi_e)_{e\in\omega}$. Thus, there exists a *universal* partial computable function f of two variables such that $f(e,n) = \Phi_e(n)$ for all $e, n \in \omega$ (note that they could both diverge). Let $\mathbf{U}$ be a Turing machine computing f, which is called a *universal Turing machine*.

A similar result holds for oracle machines or Turing functionals (see [54] and [11] for more details). Intuitively, oracle machines are Turing machines which receive and may make use of the extra information encoded in some *oracle*. An oracle machine M given oracle $B \in 2^{\leq\omega}$ is said to compute a *B-partial computable function* $\Phi_M^B :\subseteq \omega \to \omega$ as before. Similarly, there is a single oracle machine (also called $\mathbf{U}$) which, given an oracle $B \in 2^{\leq\omega}$, will enumerate all B-partial computable functions $(\Phi_e^B)_{e\in\omega}$.

If there is an oracle machine M that computes the set $A \subseteq \omega$ given an oracle B, then A is said to be *Turing reducible* to B, or *B-computable*. This is denoted by $A \leq_{\mathrm{T}} B$.

Similarly, a set A is *computably enumerable in* B (or, B*-c.e.*) whenever $A = \operatorname{dom} \Phi$ for some B-partial computable function Φ. *Relativization* is the process of generalizing notions or results from Turing machines to oracle machines given arbitrary oracle-power. Most results we cite from computability theory may be relativized appropriately.

In later sections, we will also make use of lower- or upper-semicomputability. A real number $x \in \mathbb{R}$ is *lower-semicomputable* (or *left-c.e.*) if its Dedekind left cut $\{q \in \mathbb{Q} : q < x\}$ is c.e. Similarly, a real-valued map $f : \omega \to \mathbb{R}$ is *lower-semicomputable* if its lower graph $\{(n, q) \in \omega \times \mathbb{Q} : q < f(n)\}$ is c.e. If $f : \omega \to [a, \infty)$ has a known, computable codomain which is bounded from below, we may equivalently characterize lower-semicomputability by the ability to be approximated from below by some computable *left-approximator*. That is, suppose there exists a computable map $\widehat{f} : \omega \times \omega \to [a, b]$ satisfying:

$$\widehat{f}(i, r) \leq \widehat{f}(i, r+1) \leq f(i), \quad \text{and} \quad \lim_{r\to\infty} \widehat{f}(i, r) = f(i),$$

for all $i, r \in \omega$. Then, it is straightforward to argue that the lower graph of f is computably enumerable. Conversely, from an enumeration of the lower graph of f, one may define such a left-approximator $\widehat{f}$ to f: for a given $i \in \omega$, one may approximate $f(i)$ as the largest rational q such that (i, q) has been enumerated into the lower graph of f. If no such rational has yet been approximated, we may take $\widehat{f}(i, r) = a$. Analogous definitions work for upper-semicomputability.

The definitions above are given for either subsets of the natural numbers or functions on the natural numbers, but they may be extended to the other standard collections of finitary objects discussed in Section 1.2 via a computable bijection.

1.4.2 Computable Analysis

Now, let us import some notions from computable analysis as presented in [24]. For the rest of this section, fix the ambient dimensions $m, n \in \omega$ and an oracle machine M.

Fix a map $\phi : \omega \to \mathbb{D}^m$ from the natural numbers to dyadic rationals in $\mathbb{R}^m$ satisfying $\phi(r) \in \mathbb{D}^m_r$ for all $r \in \omega$. Then, ϕ is said to be a *Cauchy representation* of $x \in \mathbb{R}^m$ if $|\phi(r) - x| \leq 2^{-r}$ for all $r \in \omega$. If ϕ is a Cauchy representation of x further satisfying $\phi(r) \leq x < \phi(r) + 2^{-r}$ for all $r \in \omega$, then ϕ is said to be a *standard Cauchy representation* for x. Such a ϕ can compute the binary expansion of x having no infinite tail of ones. For any $\phi : \omega \to \mathbb{D}^m$, we will denote by M^ϕ the partial map computed by M given ϕ encoded as an infinite binary string for its oracle. And M^x will denote the partial map computed by M^ϕ, where ϕ is some standard Cauchy representation of x.

M is said to *compute* a real function $f : \mathbb{R}^m \to \mathbb{R}^n$ if ϕ being a Cauchy representation for some $x \in \mathrm{dom}(f)$ implies M^ϕ is a Cauchy representation of $f(x)$.

Now, fix a metric space (Ω, d). Two subsets X and Y of Ω are said to be *ε-close* for $\varepsilon > 0$ if for each $x \in X$, there exists $y \in Y$ such that $d(x, y) < \varepsilon$, and vice versa. The *Hausdorff distance* between X and Y is then defined:

$$d_{\mathrm{H}}(X, Y) = \inf \{\varepsilon > 0 : X \text{ is } \varepsilon\text{-close to } Y\}.$$

We may now recall the effective versions of certain metric spaces as presented in [19].

The triple $\mathbf{\Omega} = (\Omega, d, \alpha)$ is a *computable metric space* if $\alpha = (\alpha_i)_{i\in\omega}$ is a dense sequence in (Ω, d) for which the function mapping $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is computable.

Take $\mathbf{\Omega} = (\Omega, d, \alpha)$ to be a computable metric space. If d happens to be a complete metric, call $\mathbf{\Omega}$ a *computable Polish space*. Any subset $X \subseteq \Omega$ of a computable metric space is called *computably compact* if either $X = \emptyset$ or there exists a computable map $f : \omega \to \omega$ such that for each $k \in \omega$,

$$d_{\mathrm{H}}(X, \Lambda_{f(k)}) < 2^{-k},$$

where Λ_e is the e-th collection enumerated in a standard, computable enumeration of all finite sub-collections of α. Then, the computable metric space $\mathbf{\Omega}$ is called *effectively compact* if Ω is computably compact in $\mathbf{\Omega}$.

1.4.3 Lightface Hierarchies

Kleene's *arithmetic hierarchy* assigns definability-complexity levels to subsets of the natural numbers. Any set $A \subseteq \omega$ is a *Σ^0_n-set* if there is a computable relation R on $n+1$ arguments such that

$$a \in A \iff \exists x_1 \forall x_2 \cdots Q x_n R(x_1, ..., x_n, a),$$

where the quantifiers alternate and end with Q being $\exists$ whenever n is odd, and $\forall$ when n is even. A *Π^0_n-set* is the complement of a Σ^0_n set. And a set which is both Σ^0_n and Π^0_n is called Δ^0_n. For convenience, each of Δ^0_0, Σ^0_0, and Π^0_0 are defined as containing exactly the computable sets. Moreover, a set is computably enumerable if and only if it is Σ^0_1, and computable if and only if it is Δ^0_1. This hierarchy may be relativized to any oracle by permitting the relation be Turing-reducible to it.

Suppose $S \subseteq 2^{<\omega}$ is a c.e. set of finite binary strings. Then $[S] := \bigcup_{\sigma \in S}[\sigma]$ is called a Σ^0_1-*class*, or an *effectively open* subset of Cantor space. The complement of a Σ^0_1-class is called a Π^0_1-*class*, or an *effectively closed* subset. Starting from these base classes, an analogous hierarchy to Kleene's arithmetic hierarchy forms for subsets of Cantor space. There are two ways to build higher levels of this lightface hierarchy. The first is by computable relations: a subset $U \subseteq 2^\omega$ is a Σ^0_n-*class* for $n \in \omega$ if there is a computable relation R on n arguments such that

$$\alpha \in U \iff \exists x_1 \forall x_2 \cdots Q x_n \ R(x_1, x_2, ..., \alpha \restriction x_n),$$

again by alternating quantifiers with an appropriate choice for the innermost one Q. The complement of a Σ^0_n-class is a Π^0_n-*class*. But we may equivalently conceive of the Σ^0_{n+1}-classes as uniformly-computable unions of Π^0_n-classes. This may be formalized using effective Borel codes. And a set which is both a Σ^0_n-class and a Π^0_n-class is also called a Δ^0_n-*class*. These notions may be relativized to any oracle $B \in 2^{\leq\omega}$, notated with B in the superscript: $\Sigma^{0,B}_n$, etc. The Fundamental Theorem of effective descriptive set theory states that for any $n \in \omega$, it holds that $\boldsymbol{\Sigma}^0_n = \bigcup_{B \in 2^{\leq\omega}} \Sigma^{0,B}_n$, where $\boldsymbol{\Sigma}^0_n$ denotes the corresponding level of the (boldface) Borel hierarchy from classical descriptive set theory. This motivates naming the hierarchy of Σ^0_n and Π^0_n-classes as the *lightface Borel hierarchy*.

1.5 Algorithmic Randomness

We review the tests-based approach to algorithmic randomness, all with respect to a given premeasure.

1.5.1 Weights

Definition 1.5.1. Let $\mathcal{V} \subseteq 2^{<\omega}$ be a collection of finite binary strings and ρ be a premeasure on 2^ω. Define for $\mathcal{V}$ its:

- *Direct ρ-weight*: $\mathrm{DW}_\rho(\mathcal{V}) := \sum_{\sigma \in \mathcal{V}} \rho([\sigma])$,
- *Prefix ρ-weight*: $\mathrm{PW}_\rho(\mathcal{V}) := \sup \{\mathrm{DW}_\rho(\mathcal{P}) : \mathcal{P} \subseteq \mathcal{V} \text{ is prefix-free}\}$,
- *Vehement ρ-weight*: $\mathrm{VW}_\rho(\mathcal{V}) := \inf \{\mathrm{DW}_\rho(\mathcal{W}) : \mathcal{W} \subseteq 2^{<\omega} \text{ and } [\mathcal{V}] \subseteq [\mathcal{W}]\}$.

When working with the s-dimensional Hausdorff premeasure ρ_s for some $s \geq 0$, we might simply write $\mathrm{DW}_s(\mathcal{V})$ for the *direct s-weight* of $\mathcal{V}$, and similar for the other weight notions.

For any $\mathcal{V} \subseteq 2^{<\omega}$, we defined $\widehat{\mathcal{V}}$ in Section 1.2 to include only those strings in $\mathcal{V}$ which are minimal with respect to the prefix relation. Then $\widehat{\mathcal{V}}$ is prefix-free and satisfies $[\widehat{\mathcal{V}}] = [\mathcal{V}]$. Using this fact, it is straightforward to verify that the above weight notions are ordered as follows:

$$\mathrm{VW}_\rho(\mathcal{V}) \leq \mathrm{PW}_\rho(\mathcal{V}) \leq \mathrm{DW}_\rho(\mathcal{V}).$$

These covering weights also relate to the s-dimensional Hausdorff outer measures. In particular, if $\mathcal{H}^s(X) < c$ for some dimension parameter $s \geq 0$, constant $c > 0$, and subset $X \subseteq 2^\omega$, then it follows that for every $r \in \omega$, there exists a collection of finite binary strings $S \subseteq 2^{\geq r}$ such that $X \subseteq [S]$ and $\mathrm{DW}_s(S) < c$.

1.5.2 Tests

We introduce some test notions, or effective schemes for identifying null-sets with respect to some premeasure. For the rest of the section, we will only consider upper-semicomputable premeasures on 2^ω. This restriction is made to ensure the collection of all tests of some type is uniformly-c.e. By relativizing these tests to some oracle, one could expand the class of premeasures considered to those which are upper-semicomputable in B.

Definition 1.5.2. Let ρ be an upper-semicomputable premeasure on 2^ω. Suppose $\mathcal{U} = (U_n)_{n\in\omega}$ is a uniformly-c.e. sequence of collections $U_n \subseteq 2^{<\omega}$ of finite binary strings. Then define $\mathcal{U}$ to be:

- a *Martin-Löf-ρ-test* if $\mathrm{DW}_\rho(U_n) \leq 2^{-n}$ for all $n \in \omega$,
- a *strong Martin-Löf-ρ-test* if $\mathrm{PW}_\rho(U_n) \leq 2^{-n}$ for all $n \in \omega$, and
- a *vehement Martin-Löf-ρ-test* if $\mathrm{VW}_\rho(U_n) \leq 2^{-n}$ for all $n \in \omega$.

And if $\mathcal{V} \subseteq 2^{<\omega}$ is a c.e. collection of finite binary strings, define $\mathcal{V}$ to be:

- a *Solovay-ρ-test* if $\mathrm{DW}_\rho(\mathcal{V}) < +\infty$, and
- a *strong Solovay-ρ-test* if $\mathrm{PW}_\rho(\mathcal{V}) < +\infty$.

We use $\mathcal{T}$ as a placeholder for any one of the above test notions. Once again, for any Hausdorff premeasure ρ_s with $s > 0$ being left-c.e., we simplify each of the above terms as *$\mathcal{T}$-s-tests*. And when $s = 1$, we might simply refer to them as *$\mathcal{T}$-tests*. Furthermore, any one of these test notions may be relativized to an oracle $B \in 2^{\leq\omega}$, regarding computable-enumerability as in B. We may refer to such tests as *$\mathcal{T}$-B-ρ-tests*.

A Martin-Löf-ρ-test (or ML-ρ-test) is the strictest notion of test we have defined, meaning from any Martin-Löf-ρ-test $\mathcal{U}$, one may produce a $\mathcal{T}$-ρ-test for any of the other test notions $\mathcal{T}$. In fact, such a $\mathcal{U}$ is already both a strong and vehement ML-ρ-test, and generates the collection $\mathcal{V} := \bigcup_n U_n$ which is a (strong) Solovay-ρ-test. Of course, any strong ML-ρ-test is also a vehement ML-ρ-test, and any Solovay-ρ-test is already a strong Solovay-ρ-test.

1.5.3 Covers

Tests act as effective covers of subsets of Cantor space.

Definition 1.5.3. Let ρ be an upper-semicomputable premeasure on 2^ω and $X \subseteq 2^\omega$ be a subset.

- Let $\mathcal{U} = (U_n)_{n\in\omega}$ be a (possibly strong or vehement) Martin-Löf-ρ-test. Then $\mathcal{U}$ is said to *cover* X whenever $X \subseteq \bigcap_n U_n$. Otherwise, X is said to have *passed* $\mathcal{U}$.
- Alternatively, let $\mathcal{V}$ be a (possibly strong) Solovay-ρ-test. Then $\mathcal{V}$ is said to *cover* X whenever, for each $x \in X$, there are infinitely many $r \in \omega$ for which $x \upharpoonright r \in \mathcal{V}$. Otherwise, X is said to have *passed* $\mathcal{V}$.

For a fixed test notion $\mathcal{T}$ (e.g., Martin-Löf) and upper-semicomputable premeasure ρ on 2^ω, an infinite binary string $x \in 2^\omega$ is said to be *$\mathcal{T}$-ρ-random* if $\{x\}$ passes all $\mathcal{T}$-ρ-tests. Otherwise, $\{x\}$ is covered by some such test, exhibiting itself to be (effectively) *$\mathcal{T}$-ρ-null*.

Note that if ρ is an upper-semicomputable, Hausdorff premeasure (i.e., the premeasure associated to an upper-semicomputable dimension function h as described in Section 1.3), being $\mathcal{T}$-ρ-null for any of the test notions $\mathcal{T}$ defined here implies being $\mathcal{H}^\rho$-null. Suppose, for instance, that $\mathcal{U}$ were a vehement ML-ρ-test and $\delta > 0$. Then, it is possible to show that there exists a sequence $(W_n)_{n\in\omega}$ where each $W_n \subseteq 2^{<\omega}$ consists of strings of diameter no greater than δ and is such that $[U_n] \subseteq [W_n]$ and $\mathrm{DW}_\rho(W_n) \leq 2 \cdot 2^{-n}$. If $\mathcal{V} = \bigcup_n W_n$, then

$$\sum_{\sigma\in\mathcal{V}} \mathcal{H}^\rho_\delta([\sigma]) = \sum_n \sum_{\sigma\in W_n} \mathrm{DW}_\rho([\sigma]) = \sum_n \mathrm{DW}_\rho(W_n) \leq \sum_n 2 \cdot 2^{-n} < \infty.$$

Since $\delta > 0$ was arbitrary, $\mathcal{H}^\rho$ is an outer measure, and $X \subseteq \limsup_{\sigma\in\mathcal{V}}[\sigma]$, the (first) Borel-Cantelli lemma implies $\mathcal{H}^\rho(X) = 0$. Additionally, supposing $\mathcal{V}$ were a strong Solovay-ρ-test covering X and $\delta > 0$, then we could restrict $\mathcal{V}$ to only those strings of diameter no more than δ while maintaining $\mathcal{V}$ being a strong Solovay-ρ-test covering X. Then, for each $\varepsilon > 0$, it is possible to show that there exists a prefix-free subset $\mathcal{P} \subseteq \mathcal{V}$ such that $[\mathcal{V}] \subseteq [\mathcal{P}]$ and $\mathrm{DW}_\rho(\mathcal{P}) < \varepsilon$. Then, by monotonicity, $\mathcal{H}^\rho_\delta(X) \leq \mathrm{DW}_\rho(\mathcal{P}) < \varepsilon$. And since both $\delta, \varepsilon > 0$ were arbitrary, we conclude X is $\mathcal{H}^\rho$-null.

For any upper-semicomputable premeasure ρ on 2^ω and any of the above test notions $\mathcal{T}$, it follows from the property that $\mathcal{H}^\rho$ is an outer measure that the set of non-$\mathcal{T}$-ρ-randoms is $\mathcal{H}^\rho$-null.

Weaker randomness notions come from stricter test notions. So, the class of Martin-Löf-ρ-randoms contains all the other classes of $\mathcal{T}$-ρ-randoms discussed here.

1.5.4 More Covering Notions

Let us start by reviewing some definitions originally appearing in [21] (and re-appearing slightly differently in [51]).

Definition 1.5.4. A countable collection $(X_i)_{i\in\omega}$ of sets *strongly covers* another set X if each element of X is also an element of infinitely many X_i.

Note that in Definition 1.5.3, we consider X to be covered by a Solovay-type test if that test strongly covers X in the sense of Definition 1.5.4.

Definition 1.5.5. For any (bounded) set $X \subseteq \mathbb{R}^m$ and $\delta > 0$, define,

$$N(X, \delta) := \min\{|Y| : Y \text{ is a set of } \delta\text{-balls covering } X\}.$$

The quantity $N(X, \delta)$ is exactly what appears in the definitions of the box-counting dimensions, as it captures the least number of δ-balls required to cover X (see [12]). This fractal dimension always bounds the Hausdorff dimension from above.

Definition 1.5.6. Fix $m \in \omega$, and let $s, \delta > 0$ and $C \geq 1$. A finite set $P \subseteq \mathbb{R}^m$ is called a (C, δ, s)*-set* if for any $x \in \mathbb{R}^m$ and $\delta \leq \varepsilon \leq 1$,

$$|P \cap B_\varepsilon(x)| \leq C \cdot \left(\frac{\varepsilon}{\delta}\right)^s.$$

Suppose that $X \subseteq \mathbb{R}^m$ has $\dim_{\mathrm{H}} X = s$. Then any (C, δ, s)-set will have no more points in X than a constant times what any δ-net on X would. That is, (C, δ, s)-sets are spread out such that they never dedicate many points to sets of small Hausdorff dimension.

Note that it is not vital that we use r-balls in these definitions for $N(X, \delta)$ or (C, δ, s)-sets. For instance, replacing these with dyadic cubes of comparable side-lengths would produce roughly equivalent notions.

Over Euclidean space, there is a relationship between Hausdorff dimension and (C, δ, s)-sets. For instance, a set of Hausdorff dimension less than s will have a (C, δ, s)-set whose δ-neighborhood strongly covers the set. The precise result originally appeared in [21] by N. Katz and T. Tao, but was simplified and slightly generalized by T. Orponen in [51].

Lemma 1.5.7 (Lemma 7.5 of [21]; Lemma 2.1 of [51])**.** *Let $0 < s \leq m$ and let $X \subseteq \mathbb{R}^m$ be a subset with $\dim_{\mathrm{H}} X < s$. Then, there exists a constant $C \geq 1$ depending only on m, s, and $\dim_{\mathrm{H}} X$ such that: for every $k \in \omega$, there exists a $(Ck^2, 2^{-k}, s)$-set P_k such that the sequence $(B_{C_m \cdot 2^{-k}}(P_k))_{k\in\omega}$ strongly covers X, where $C_m \geq 1$ only depends on m.*

The above lemma was leveraged by Orponen to generalize a weak form of the Marstrand-Mattila Projection Theorem for arbitrary subsets of Euclidean space [51] (see Theorem 5.1.2). We will present an effective proof of the Katz-Tao lemma later in Section 5.1.

Also recall the related notion of an *optimal cover* originally by J. Miller [47].

Definition 1.5.8. Let ρ be a premeasure on 2^ω and $S \subseteq 2^{<\omega}$. Then a *ρ-optimal cover* of S is a set $S^* \subseteq 2^{<\omega}$ satisfying $[S] \subseteq [S^*]$ and $\mathrm{DW}_\rho(S^*) = \mathrm{VW}_\rho(S)$.

Optimal covers witness the vehement ρ-weight of a set. In general, it is unclear whether optimal covers exist. However, a result by P. Hudelson proves a necessary condition for the existence of optimal covers.

Lemma 1.5.9 (Theorem 3.4.22 of [18])**.** *Let ρ be a convex Hausdorff premeasure on 2^ω. Then, for all $S \subseteq 2^{<\omega}$, there exists a ρ-optimal cover of S.*

In particular, if ρ is also computable and S is c.e., then one may find another c.e. set A (uniformly in ρ and S) such that $\widehat{A}$ is a ρ-optimal cover of S.

Essential to the proof of Lemma 1.5.9 is the fact that whenever ρ is convex and $\sigma \in 2^{<\omega}$, then $\{\sigma\}$ is the DW_ρ-minimal cover of $[\sigma]$ among all other collections $S \subseteq [\![\sigma]\!]$ covering $[\sigma]$.

Observe that an optimal cover will necessarily be prefix-free by minimality. When $\rho = \rho^s$, simplify the term to an *s-optimal cover*. Optimal covers were originally employed by Miller to construct a Δ^0_2-definable real with effective dimension equal to $1/2$ yet unable to compute any real of greater effective dimension [47]. We will make use of optimal covers for effectivizing Lemma 1.5.7 in Section 5.1.

1.6 Prefix Complexity

1.6.1 Incompressibility

Recall the universal oracle machine $\mathbf{U}$ from Section 1.4. The universality of $\mathbf{U}$ implies some invariance and minimality properties in the following senses. Let σ and τ be finite binary strings and M to be an oracle Turing machine on finite binary strings. By universality, it holds that there exists a string ν_M which may be prepended to the input σ to allow $\mathbf{U}$ to simulate M on σ given any oracle B, i.e., $\mathbf{U}^B(\nu_M \frown \sigma) = M^B(\sigma)$. Define the *conditional plain Kolmogorov complexity of σ given τ in M* to be

$$C_M(\sigma \mid \tau) = \min\left\{\mathrm{len}(\pi) : M^{\overline{\tau}}(\pi) \downarrow= \sigma\right\},$$

or $+\infty$ when minimizing over an empty set; where we map the condition τ into a prefix-free set of strings via $\overline{\tau} = \tau_0\tau_0\tau_1\tau_1 \cdots \tau_{\mathrm{len}(\tau)-1}\tau_{\mathrm{len}(\tau)-1}01$ (on account of the fact that the set of oracles on which a machine M may converge on a fixed input σ must be prefix-free; see [11] for more details). Then, call $C(\sigma \mid \tau) := C_{\mathbf{U}}(\sigma \mid \tau)$ the *conditional plain Kolmogorov complexity of σ given τ*. Under any other choice of universal machine $\mathbf{U}$, this quantity is invariant up to an additive constant independent of σ and τ. Conditional plain complexity also satisfies minimality, in that for any M:

$$C(\sigma \mid \tau) \leq C_M(\sigma \mid \tau) + O_M(1).$$

for some constant term depending on M. The *plain Kolmogorov complexity* of a string σ is simply $C(\sigma) = C(\sigma \mid \langle\rangle)$. We call any finitary object π satisfying $M^{\overline{\tau}}(\pi) \downarrow= \sigma$ an *M-description* for "$\sigma \mid \tau$". Any M-description π of $\sigma \mid \tau$ for which $\mathrm{len}(\pi) = C_M(\sigma \mid \tau)$ is also called *minimal*.

A *prefix-free (PF) oracle machine* is an oracle machine on finite binary strings with prefix-free domain for any oracle. Any PF oracle machine M computes a prefix-free partial function $\Psi_M :\subseteq 2^{<\omega} \to 2^{<\omega}$ as before. Again, by the Enumeration Theorem, there exists a *universal* PF oracle machine $\mathbf{U}_{\mathrm{PF}}$. That is, for each PF oracle machine M, there exists a coding string $\nu_M \in 2^{<\omega}$ such that given any oracle B and input $\sigma \in 2^{<\omega}$, we have

$\mathbf{U}_{\text{PF}}^{B}(\nu_M {}^\frown\sigma) = M^B(\sigma)$. Once again, $\mathbf{U}_{\text{PF}}$ admits an invariant and minimal complexity notion in the sense that for any other PF oracle machine M, there exists an additive constant term $O_M(1)$ such that for any $\sigma, \tau \in 2^{<\omega}$,

$$C_{\mathbf{U}_{\text{PF}}}(\sigma \mid \tau) \leq C_M(\sigma \mid \tau) + O_M(1).$$

We let $K(\sigma \mid \tau) = C_{\mathbf{U}_{\text{PF}}}(\sigma \mid \tau)$ denote the *conditional prefix Kolmogorov complexity of σ given τ*. Both plain and prefix complexities may be relativized to any oracle by pairing the oracle and the condition string (see [11] for more details). Then $K^B(\sigma \mid \tau) := C^B_{\mathbf{U}_{\text{PF}}}(\sigma \mid \tau)$ denotes the *conditional prefix Kolmogorov complexity of σ given τ relative to B*.

These complexity notions may also be extended to any other standard collection of finitary objects $\mathscr{N}$ with a constant amount of overhead, meaning we would only need to sacrifice at most a constant number of bits to describe the computable bijection between $2^{<\omega}$ and $\mathscr{N}$.

1.6.2 Results on Prefix Complexity

We review some standard results on prefix complexity.

First, one might refer to the following result as either the *symmetry of information* or the *chain rule* for conditional prefix Kolmogorov complexity. It was first shown by P. Gács [14]. The first equation is the strong form and is a property specific to prefix complexity; whereas the second equation is a weak form (where equality holds up to logarithmic terms in the string lengths) shared by many Kolmogorov complexity notions.

Theorem 1.6.1 (Chain Rule for Conditional Prefix Kolmogorov Complexity; Theorem 1 of [14])**.** *There exists a constant $c \in \omega$ such that for any $\sigma, \tau, \zeta \in 2^{<\omega}$,*

$$\begin{aligned} K(\sigma, \tau \mid \zeta) &= K(\sigma \mid \zeta) + K(\tau \mid \sigma, K(\sigma), \zeta) \pm c \\ &= K(\sigma \mid \zeta) + K(\tau \mid \sigma, \zeta) \pm [O(\log \operatorname{len}(\sigma)) + c]. \end{aligned}$$

An easy consequence of the chain rule is the *subadditivity* of K: i.e., for any $\sigma, \tau \in 2^{<\omega}$,

$$K(\sigma^\frown\tau) \leq K(\sigma, \tau) \leq K(\sigma) + K(\tau) + O(1).$$

We may also say something about the growth-rate of K as an integer function. Write $\operatorname{len}(n) + \operatorname{len}(\operatorname{len}(n)) + \cdots$ for the sum of nested applications of len, including only those terms which are positive. Let $\log^*(n)$ denote the number of such terms in that sum.

Thenm, for any natural number $n \in \omega$, the optimized prefix-codes in equation (3.2) of [29] demonstrate

$$\begin{aligned} K(n) &\leq \log^* n + \operatorname{len}(n) + \operatorname{len}(n)) + \operatorname{len}(\operatorname{len}(\operatorname{len}(n))) + \cdots + O(1) \\ &\leq \log(1+n) + 2\log\log(2+n) + O(1) \\ &= O(\log n) + O(1). \end{aligned} \tag{1.2}$$

Intuitively, we consider n to be a finite binary string of length $\operatorname{len}(n)$. One description of n involves storing an index (of length no more than $\operatorname{len}(n)$) of where n appears among all other strings of length $\operatorname{len}(n)$ when ordered lexicographically. This algorithm requires a description for $\operatorname{len}(n)$ to work, so we might iterate the argument on $\operatorname{len}(n)$ as appearing somewhere in the list of all strings of length $\operatorname{len}(\operatorname{len}(n))$, and so on. The $\log^* n$ term is necessary because our machine must be prefix-free.

Together, the chain rule 1.6.1 and the inequality in (1.2) imply the results of Example 3.1.6 of [29]. Namely, that for any finitary $n \in \omega$,

$$\begin{aligned} K(n) &\leq K(n \mid \operatorname{len}(n)) + K(\operatorname{len}(n)) + O(1) \\ &\leq K(n \mid \operatorname{len}(n)) + \log^* \operatorname{len}(n) + \operatorname{len}(\operatorname{len}(n)) + \cdots + O(1) \\ &\leq K(n \mid \operatorname{len}(n)) + \log(1+\operatorname{len}(n)) + 2\log\log(2+\operatorname{len}(n)) + O(1) \\ &= K(n \mid \operatorname{len}(n)) + O(\log \operatorname{len}(n)) + O(1). \end{aligned} \tag{1.3}$$

This fact extends to a complexity bound for any integer tuple based on its Euclidean length (see Observation A.3 of [34]): for any tuple-length $m \in \omega$ and integer tuple $z \in \mathbb{Z}^m$,

$$K(z) \leq m \cdot \log(1 + ||z||) + 2 \cdot \log\log(2 + ||z||) + O_m(1). \tag{1.4}$$

The form of this inequality extends to any r-dyadic rational tuple $q \in \mathbb{D}_r^m$, where the constant term may depend on both m and r.

We include here the conditional version of another standard observation for Kolmogorov complexity: that the complexity of the output of an algorithm is bounded from above by the the complexity of the input plus some overhead to describe the algorithm. This follows from the universality of $\mathbf{U}_{\mathrm{PF}}$.

Theorem 1.6.2 (Conditional version of Proposition 3.5.4 of [11])**.** *Let $\Phi :\subseteq 2^{<\omega} \to 2^{<\omega}$ be partial computable. Then, there exists a constant $c \in \omega$ such that for any $\sigma \in \operatorname{dom}(\Phi)$*

and $\tau \in 2^{<\omega}$,

$$K(\Phi(\sigma) \mid \tau) \leq K(\sigma \mid \tau) + c.$$

As a consequence, as noted on page 218 of [29], one may conclude that K is continuous as an integer function. That is, for any natural numbers $n, \Delta n \in \omega$,

$$|K(n + \Delta n) - K(n)| \leq K(\Delta n) + O(1).$$

We might extend this and Theorem 1.6.2 as follows. Define $\Phi :\subseteq 2^{<\omega} \to 2^{<\omega}$ to be *partial co-computable (p.c.c.)* if Φ is injective and has a partial computable inverse defined on the range of Φ. And Φ is called *partial bi-computable (p.b.c.)* if Φ is both partial computable and partial co-computable.

Proposition 1.6.3. *Let* $\Phi :\subseteq 2^{<\omega} \to 2^{<\omega}$*. Then there exists a constant* $c \in \omega$ *such that for any* $\sigma \in \operatorname{dom}(\Phi)$ *and* $\tau \in 2^{<\omega}$,

$$\begin{aligned}
\Phi \textit{ is p.c.} &\implies K(\Phi(\sigma) \mid \tau) \leq K(\sigma \mid \tau) + c \quad \textit{and} \quad K(\tau \mid \sigma) \leq K(\tau \mid \Phi(\sigma)) + c,\\
\Phi \textit{ is p.c.c.} &\implies K(\sigma \mid \tau) \leq K(\Phi(\sigma) \mid \tau) + c \quad \textit{and} \quad K(\tau \mid \Phi(\sigma)) \leq K(\tau \mid \sigma) + c,\\
\Phi \textit{ is p.b.c.} &\implies K(\Phi(\sigma) \mid \tau) = K(\sigma \mid \tau) \pm c \quad \textit{and} \quad K(\tau \mid \Phi(\sigma)) = K(\tau \mid \sigma) \pm c.
\end{aligned}$$

Let us describe a general scheme for proving bounds for prefix complexity. Suppose we would like to prove a bound of the form:

$$K(a \mid b) \leq K(a_1 \mid b_1) + \cdots + K(a_k \mid b_k) + O(1),$$

where a, b and $a_1, b_1, ..., a_k, b_k$ are some finitary arguments and $O(1)$ may depend on k. In plainer language, this would read as "one may produce a description for a given b and given descriptions for each a_i given b_i."

Formally, to prove a bound of this form, one should construct an appropriate PF oracle machine exhibiting it. The machine M should interpret its input as a tuple of k strings: $\langle \pi_{\widehat{a}_1|\widehat{b}_1}, ..., \pi_{\widehat{a}_k|\widehat{b}_k} \rangle$, and read a string $\widehat{b}$ from its oracle tape. The machine may use $\widehat{b}$ and any other results stored along the course of the computation to produce some strings called $\widehat{b}_i$ for any of the $1 \leq i \leq k$. Also, for any $1 \leq i \leq k$ for which $\widehat{b}_i$ is defined, the machine may also simulate $\mathbf{U}_{\mathrm{PF}}$ with oracle $\widehat{b}_i$ and input $\pi_{\widehat{a}_i|\widehat{b}_i}$ to compute and report a string called $\widehat{a}_i$ if the computation converges. Of course, it depends on the specific argument how exactly the machine will operate.

With this oracle machine now defined, one would pass into M the oracle b and the input consisting of minimal $\mathbf{U}_{\mathrm{PF}}$-descriptions $\pi_{a_i|b_i}$ for a_i given b_i for each $1 \leq i \leq k$. Supposing that M was properly defined so as to halt on this choice of oracle and input and to correctly report a, we may then conclude:

$$\begin{aligned} K(a \mid b) &\leq C_M^b(a_1, ..., a_k \mid b_1, ..., b_k, k) + O(1) \\ &\leq \mathrm{len}(\pi_{a_1|b_1}) + \cdots + \mathrm{len}(\pi_{a_k|b_k}) + K(k) + O(1) \\ &= K(a_1 \mid b_1) + \cdots + K(a_k \mid b_k) + K(k) + O(1). \end{aligned}$$

It is standard practice to instead verbally explain an (intuitive) algorithm which produces a from b and the other descriptions $a_i \mid b_i$, and then to appeal to the Church-Turing thesis to conclude there exists a prefix-free oracle machine simulating that process. Several of the proofs provided here follow this form for the sake of clarity.

One may use this scheme to prove Proposition 1.6.3. It also works for *two-part descriptions*. Let $\mathscr{N}$ be any standard collection of finitary objects and $n \in \mathscr{N}$. Given $n \in A$ for some finite set $A \subset \mathscr{N}$, we have that $K(n) \leq K(A) + \log |A| + O(1)$. That is, a description (of a computable enumeration) of A, together with a description for the index i when n appears in that enumeration of A, suffice to describe n as the i-th element enumerated into A. In this sense, the pair (A, i) forms a two-part description for n. Two-part descriptions are also important in the study of algorithmic statistics [71].

An easy consequence of $\mathbf{U}_{\mathrm{PF}}$ being prefix-free is the *Kraft Inequality*.

Theorem 1.6.4 (Kraft Inequality, Proposition 3.7.1 of [11])**.**

$$\sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)} \leq 1.$$

Next, Chaitin's *Counting Theorem* offers useful bounds for the behavior of K on the strings of some fixed length. This essentially follows from the universality of $\mathbf{U}_{\mathrm{PF}}$.

Theorem 1.6.5 (Chaitin's Counting Theorem [8], Theorem 3.7.6 of [11])**.** *There exists a constant $c \in \omega$ such that for any $n, r \in \omega$,*

(i) $\max \{K(\sigma) : \sigma \in 2^n\} = n + K(n) \pm c$, *and*

(ii) $|\{\sigma \in 2^n : K(\sigma) \leq n + K(n) - r\}| \leq 2^{n-r+c}$.

It is interesting to note that the theory of prefix complexity (an algorithmic entropy notion) mimics that of Shannon entropy [26], and that K satisfies the same linear inequalities true for all inputs as Shannon entropy H up to a logarithmic term in the sum of the lengths of the strings involved. This may be made precise by translating between random variables with finite range and finite binary strings, as is done in [15].

1.7 Lifting Prefix Complexity to Reals

Prefix complexity may be lifted to infinitary objects via finitary approximations. We state the following results how they were originally presented: over Euclidean space. Analogous statements also hold over Cantor space.

1.7.1 Lifting Conditional Complexity

Take any ambient dimensions $m, n \in \omega$, points $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, and precision-levels $r, s \in \omega$. A straightforward definition of the complexity of x given y up to these precision-levels would be to evaluate the complexity: $K(x \upharpoonright r \mid y \upharpoonright s)$.

An alternative lift comes from the intuition offered by A. Shen and N. Vereshchagin in [59]: that for rationals $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$, the conditional complexity $K(p \mid q)$ could be understood as the complexity of the problem $Q \to P$, where P is the problem "construct p" and Q is the problem "construct q." So, the lift of conditional complexity to arbitrary subsets $X \subseteq \mathbb{R}^m$ given $Y \subseteq \mathbb{R}^n$ should reflect the complexity of constructing a point $p \in X \cap \mathbb{Q}^m$ from any point $q \in Y \cap \mathbb{Q}^n$. This definition was made explicit by J. Lutz and N. Lutz in [34].

In particular, they defined the *conditional prefix complexity of X given Y* to be:

$$K\left(X \mid Y\right) := \max_{q} \left\{ \min_{p} \left\{ K\left(p \mid q\right) : p \in X \cap \mathbb{Q}^m \right\} : q \in Y \cap \mathbb{Q}^n \right\},$$

or $+\infty$ if either set is empty. Notice that this notion of complexity is governed only by the rational elements of the sets involved. Unconditionally, the *prefix complexity of X* is:

$$K\left(X\right) := \min\left\{K\left(p\right) : p \in X \cap \mathbb{Q}^m\right\},$$

or $+\infty$ whenever X contains no rationals. Thus, a set is as simple as its simplest, rational element.

Now, for any $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, the *conditional prefix complexity of x to precision-*

level r given y to precision-level s is:

$$K_{r|s}(x \mid y) := K\left(B_{2^{-r}}(x) \mid B_{2^{-s}}(y)\right),$$

while the *prefix complexity of x to precision-level r (possibly given $q \in \mathbb{Q}^n$)* is:

$$K_r(x \mid q) := K(B_{2^{-r}}(x) \mid \{q\}), \text{ and}$$
$$K_r(x) := K(B_{2^{-r}}(x)).$$

In the coming subsection, we will compare $K_{r|s}(x \mid y)$ to $K(x \upharpoonright r \mid y \upharpoonright s)$.

It will also serve us to identify rational elements of a subset which witness the set's prefix complexity. We borrow the nomenclature of A Case. and J. Lutz [6].

Definition 1.7.1. Let $X \subseteq \mathbb{R}^m$, and suppose $x \in X \cap \mathbb{Q}^m$ satisfies $K(x) \leq K(X) + \varepsilon$ for some $\varepsilon \geq 0$. We say that x is an *ε-approximate K-minimizer* of X. And if this holds for $\varepsilon = 0$, then x is a *K-minimizer* of X.

1.7.2 Results on Lifted Conditional Complexity

We recall two lemmas about $K_{r|s}$ known as its *linear sensitivities.*

Lemma 1.7.2 (Linear Sensitivity of K_r in r; Lemma 3.8 of [6])**.** *For all $m \in \omega$, there exists $c \in \omega$ such that for all $x \in \mathbb{R}^m$ and $r, \Delta r \in \omega$,*

$$K_r(x) \leq K_{r+\Delta r}(x) \leq K_r(x) + K(r) + m\Delta r + a_{\Delta r} + c,$$

where $a_{\Delta r} := K(\Delta r) + 2\log\left(\left\lceil \frac{1}{2}\log m \right\rceil + \Delta r + 3\right) + \left(\left\lceil \frac{1}{2}\log m\right\rceil + 3\right) m + K(m) + 2\log m.$

Lemma 1.7.3 (Linear Sensitivity of $K_{r|s}$ in s; Lemmas 7 and 8 of [34])**.** *For all $m, n \in \omega$, there exists $c \in \omega$ such that for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $q \in \mathbb{Q}^n$, and $r, s, \Delta r, \Delta s \in \omega$,*

(i) $K_r(x \mid q) \leq K_{r+\Delta r}(x \mid q) \leq K_r(x \mid q) + m\Delta r + 2\log(1 + \Delta r) + K(r, \Delta r) + c;$

(ii) $K_{r|s}(x \mid y) \geq K_{r|s+\Delta s}(x \mid y) \geq K_{r|s}(x \mid y) - n\Delta s - 2\log(1 + \Delta s) + K(s, \Delta s) + c.$

A consequence of the linear sensitivities is that one may approximate the conditional prefix complexity of Euclidean points more simply via dyadic truncations.

Lemma 1.7.4 (Lemma A.1 of [40])**.** *For all $m, n \in \omega$, there exists $c \in \omega$ such that for all $x \in \mathbb{R}^m$, $q \in \mathbb{Q}^n$, and $r \in \omega$,*

$$|K_r(x \mid q) - K(x \upharpoonright r \mid q)| \leq K(r) + c.$$

Lemma 1.7.5 (Lemma A.3 of [40])**.** *For all $m, n \in \omega$, there exists $c \in \omega$ such that for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \omega$,*

$$\left|K_{r|s}(x \mid y) - K(x \upharpoonright r \mid y \upharpoonright s)\right| \leq K(r) + K(s) + c.$$

One may then conclude an approximate symmetry of information for prefix complexity on Euclidean space. Again, this is a weak form of symmetry of information.

Theorem 1.7.6 (Approximate Chain Rule; Lemma 4 of [40])**.** *For any $m, n \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r \geq s \in \omega$,*

(i) $\left|K_{r|r}(x \mid y) + K_r(y) - K_r(x, y)\right| \leq O_{m,n}(\log r) + O_n(\log\log ||y||) + O_{m,n}(1)$;

(ii) $\left|K_{r|s}(x \mid x) + K_s(x) - K_r(x)\right| \leq O_m(\log r) + O_m(\log\log ||x||) + O_m(1)$.

We note that when computing with infinitary objects such as infinite binary strings or Euclidean reals, one must distinguish between *oracle* and *conditional* accesses. While oracle access permits computations with approximations of the given data to arbitrary precisions, conditional access places a firm limit on the precision-level to which one may access the given data.

Lemma 1.7.7 (Lemma 14 of [34])**.** *For all $m, n \in \omega$, there exists a constant $c \in \omega$ such that, for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \omega$,*

$$K_r^y(x) \leq K_{r|s}(x \mid y) + K(s) + c.$$

1.8 Semimeasures

This section presents an alternative approach to measuring the information-content or algorithmic-complexity of finitary objects through mass distribution.

1.8.1 Discrete Semimeasures

A function $m : 2^{<\omega} \to [0,1]$ is said to be a *discrete semimeasure* on 2^ω if it satisfies $\sum_{\sigma \in 2^{<\omega}} m(i) \leq 1$. There exists a (reference) *optimal* lower-semicomputable discrete semimeasure $\mathbf{m}$ on 2^ω. That is, if m is any other lower-semicomputable discrete semimeasure, there exists a constant $\beta > 0$ such that for all $\sigma \in 2^{<\omega}$, we have $\mathbf{m}(\sigma) \geq \beta \cdot m(\sigma)$. Refer to Section 4.3 of [29] for more details.

There is a natural connection between prefix complexity and lower-semicomputable discrete semimeasures. Levin originally introduced the concept of an *information content measure*, or an upper-semicomputable, partial function $F :\subseteq 2^{<\omega} \to \omega$ satisfying $\sum_{\sigma \in 2^{<\omega}} 2^{-F(\sigma)} \leq 1$, as described in [11]. By the Kraft Inequality (1.6.4) and the universality of $\mathbf{U}_{\mathrm{PF}}$, K is considered to be minimal among the the information content measures. Similarly, $2^{-K(\cdot)}$ is maximal (i.e., optimal) among the lower-semicomputable discrete semimeasures.

This relationship is made more precise in *Levin's Coding Theorem*. We start by reviewing two classes of lower-semicomputable discrete semimeasures on $2^{<\omega}$.

Definition 1.8.1. Fix an oracle machine M on $2^{<\omega}$, and $\sigma \in 2^{<\omega}$.

- Let $R_M(\sigma) := 2^{-C_M(\sigma)}$ denote the *algorithmic probability* of σ under M.
- Let $Q_M(\sigma) := \sum_{M(\pi)\downarrow=\sigma} 2^{-\operatorname{len}(\pi)}$ denote the *a priori probability* of σ under M.

When M is the reference universal PF oracle machine, simply write $R \equiv R_{\mathbf{U}_{\mathrm{PF}}}$ and $T \equiv T_{\mathbf{U}_{\mathrm{PF}}}$.

Levin proved that $\mathbf{m}$, R, and Q all capture essentially the same quantity.

Theorem 1.8.2 (Levin's Coding Theorem, Theorem 3.9.4 of [11])**.** *For any* $\sigma \in 2^{<\omega}$,

$$K(\sigma) = -\log R(\sigma) = -\log Q(\sigma) \pm O(1) = -\log \mathbf{m}(\sigma) \pm O(1).$$

One informal corollary of the Coding Theorem is that: having many *long* $\mathbf{U}_{\mathrm{PF}}$-descriptions implies having at least one *short* $\mathbf{U}_{\mathrm{PF}}$-description. To see this, suppose Π is a collection of strings $\pi \in 2^{<\omega}$ all of a common length $\ell \in \omega$ and serving as $\mathbf{U}_{\mathrm{PF}}$-descriptions for the string $\sigma \in 2^L$. Then, the *a priori* probability of σ will be at least:

$$Q(\sigma) = \lambda([\{\pi \in 2^{<\omega} : \mathbf{U}_{\mathrm{PF}}(\pi) \downarrow= \sigma\}]) \geq \lambda([\Pi]) = \sum_{\pi \in \Pi} 2^{-\operatorname{len}(\pi)} = |\Pi| \cdot 2^{-\ell}.$$

Thus, by the Coding Theorem 1.8.2, σ has an appropriately shorter $\mathbf{U}_{\mathrm{PF}}$-description:

$$K(\sigma) \leq -\log Q(\sigma) + O(1) \leq \ell - \log |\Pi| + O(1).$$

1.8.2 Continuous Semimeasures

Discrete semimeasures do not take into account the topology of Cantor space, while a continuous semimeasure does by considering how mass distributes from a string to its extensions. In particular, a function $M : 2^{<\omega} \to [0,1]$ is said to be a *continuous semimeasure* on 2^ω if it satisfies $M(\sigma) \geq M(\sigma^\frown 0) + M(\sigma^\frown 1)$ for all $\sigma \in 2^{<\omega}$. As in the discrete case, there exists a reference *optimal* lower-semicomputable continuous semimeasure $\mathbf{M}$ on 2^ω. That is, if M is any other lower-semicomputable continuous semimeasure, there exists a constant $\beta > 0$ such that for all $\sigma \in 2^{<\omega}$, we have $\mathbf{M}(\sigma) \geq \beta \cdot M(\sigma)$. Refer to Section 4.5 of [29] for more details.

We denote by $\mathrm{KM}(\sigma) := -\log \mathbf{M}(\sigma)$ the *a priori complexity* of a string $\sigma \in 2^{<\omega}$. It is interesting to compare this complexity notion to prefix complexity. There are few satisfying bounds on their difference. Refer to [70] and Section 4.5.5 of [11] for more details.

It is important to mention the fundamental role that constructive martingales have played in the development of algorithmic information theory. By 2000, J. Lutz had introduced algorithmically-restricted betting strategies called *constructive supergales* as a generalization to the constructive martingales in C. Schnorr's early work on algorithmic randomness [31, 58]. Let us see why, for a certain class of premeasures, these betting strategies are interchangeable with the lower-semicomputable continuous semimeasures.

Given a premeasure ρ on 2^ω, a *ρ-supergale* is a map $\delta : 2^{<\omega} \to [0, +\infty)$ such that

$$\delta(\sigma) \cdot \rho([\sigma]) \geq \delta(\sigma^\frown 0) \cdot \rho([\sigma^\frown 0]) + \delta(\sigma^\frown 1) \cdot \rho([\sigma^\frown 1]),$$

for each $\sigma \in 2^{<\omega}$. If indeed $\rho = \rho_s$ for some $s \geq 0$, simply refer to δ as an *s-supergale*.

A ρ-supergale is said to *succeed* on a subset $X \subseteq 2^\omega$ if for each $x \in X$, we have $\limsup_{r\to\infty} \delta(x \upharpoonright r) = +\infty$. Thus, to succeed on an infinite sequence $x \in 2^\omega$ is to earn arbitrarily much by taking bets on new bits of x following the betting strategy δ.

It had already been shown by J. Lutz in [30] that the Hausdorff dimension of a subset $X \subseteq 2^\omega$ may be characterized by the ability for s-supergales to succeed on X. That is,

$$\dim_{\mathrm{H}}(X) = \inf \{s \geq 0 : \text{there exists an } s\text{-supergale succeeding on } X\}. \tag{1.5}$$

Lutz's subsequent idea was to restrict supergales by an effectivity condition and see whether they could still recover classical Hausdorff dimension. Call a ρ-supergale to be *constructive* if it is lower-semicomputable. If ρ is length-invariant, strictly-positive, and computable on the cylinder sets $\mathcal{B}$ (e.g., any s-dimensional Hausdorff premeasure ρ_s with computable $s \geq 0$), it holds that δ is a (constructive) ρ-supergale if and only if the map $M : \sigma \mapsto \delta(\sigma) \cdot \rho([\sigma])$ is a (lower-semicomputable) continuous semimeasure.

Thus, continuous semimeasures are also related to Hausdorff dimension in 2^ω, and we may extend the notion of success to them as follows: a continuous semimeasure M is said to *ρ-succeed* on $X \subseteq 2^\omega$ if for each $x \in X$, we have $\limsup_{r\to\infty} M(x \upharpoonright r)/\rho([x \upharpoonright r]) = +\infty$. If M is some lower-semicomputable continuous semimeasure which ρ-succeeds on X, then, by optimality, $\mathbf{M}$ also ρ-succeeds on X. When $\rho = \rho_s$, we refer to this as *s-success*.

From this perspective, the language of constructive supergales ports into the language of lower-semicomputable continuous semimeasures.

Another characterization of s-success for lower-semicomputable continuous semimeasures has to do with the existence of strong Solovay-s-tests of arbitrarily small prefix s-weight. Call $(q_n)_{n\in\omega} \subseteq \mathbb{Q}_{>0}$ *rapidly decreasing* if $\lim_{n\to\infty} q_n \cdot 2^n = 0$.

Theorem 1.8.3. *Fix $X \subseteq 2^\omega$ and computable $s > 0$. Then the following are equivalent.*

(i) There is a uniformly-c.e. sequence of strong Solovay-s-tests $(\mathcal{V}_n)_n$ and a computable, rapidly decreasing sequence $(q_n)_n$ where each $\mathcal{V}_n$ covers X and satisfies $\mathrm{PW}_s(\mathcal{V}_n) \leq q_n$*;*

(ii) Some constructive s-supergale succeeds on X;

(iii) Some lower-semicomputable continuous semimeasure s-succeeds on X; and

(iv) $\mathbf{M}$ *s-succeeds on X.*

Proof. It suffices to prove (i) $\iff$ (iii). First, suppose (i). For each $n \in \omega$, the map

$$M_n(\sigma) := \frac{1}{q_n} \mathrm{PW}_s(\{\tau \in \mathcal{V}_n : \sigma \preceq \tau\})$$

is a lower-semicomputable continuous semimeasure. And, whenever $\sigma \in \mathcal{V}_n$, it holds that $M_n(\sigma) \geq \frac{1}{q_n} \cdot 2^{-s\cdot\mathrm{len}(\sigma)}$. Now, define another lower-semicomputable continuous semimeasure

$$M(\sigma) := \sum_{n\in\omega} \frac{M_n(\sigma)}{2^{n+1}}.$$

Then, for each $\sigma \in \mathcal{V}_n$, we have $M(\sigma) \geq \frac{1}{q_n} \cdot 2^{-(n+1)-s\cdot\mathrm{len}(\sigma)}$. Now, if $x \in X$ and X is covered by each $\mathcal{V}_n$, it holds that for each $n \in \omega$, there exist infinitely many $r \in \omega$ for which $x \upharpoonright r \in \mathcal{V}_n$. Hence,

$$\limsup_{r\to\infty} \frac{M(x \upharpoonright r)}{2^{-s\cdot \mathrm{len}(x\upharpoonright r)}} = \limsup_{r\to\infty} M(x \upharpoonright r) \cdot 2^{s\cdot r} \geq \frac{1}{q_n} \cdot 2^{-(n+1)} \to +\infty$$

as $n \to +\infty$ since $(q_n)_n$ is rapidly decreasing. Thus, M s-succeeds on X.

In the other direction, suppose (iii). Fix a computable, rapidly decreasing sequence $(q_n)_n$ and define for each $n \in \omega$ the c.e. set:

$$\mathcal{V}_n := \left\{\sigma \in 2^{<\omega} : M(\sigma) \cdot 2^{s\cdot \mathrm{len}(\sigma)} \geq \frac{1}{q_n}\right\}.$$

Notice that if $\mathcal{P} \subseteq \mathcal{V}_n$ is prefix-free, then

$$\mathrm{DW}_s(\mathcal{P}) = \sum_{\sigma\in\mathcal{P}} 2^{-s\cdot \mathrm{len}(\sigma)} \leq q_n \cdot \sum_{\sigma\in\mathcal{P}} M(\sigma) \leq q_n,$$

so $\mathrm{PW}_s(\mathcal{V}_n) \leq q_n$ for all $n \in \omega$. Finally, for a fixed $n \in \omega$, if $x \in X$, then M s-succeeding on X implies there are infinitely many $r \in \omega$ for which $M(x \upharpoonright r) \cdot 2^{s\cdot r} \geq \frac{1}{q_n}$, or $x \upharpoonright r \in \mathcal{V}_n$. So, each $\mathcal{V}_n$ is a strong Solovay-s-test covering X, and the sequence $(\mathcal{V}_n)_n$ is uniformly-c.e. □

1.9 Effective Hausdorff Dimension

J. Lutz was the first to introduce a refinement to Hausdorff dimension in the form of *constructive dimension* [31], which could distinguish between subsets of zero Hausdorff dimension by their algorithmically-exploitable patterns. Shortly after, E. Mayordomo demonstrated this quantity matched an algorithmic information density involving Kolmogorov complexity first studied by L. Staiger [45, 63]. In this section, we review a few of the characterizations for effective dimension, but not necessarily in their historical order.

For simplicity, we define the effective dimension notions in this section over 2^ω. With slight adjustments, some may be extended to any $\mathbb{R}^m$.

Inspired by the definition of the local dimension of an outer measure in K. Falconer's book [12], define local dimension for outer measures and semimeasures as follows.

Definition 1.9.1. Let μ either be a discrete semimeasure, continuous semimeasure, or outer measure on 2^ω. Then the *(lower) local dimension* of a subset $X \subseteq 2^\omega$ with respect to μ is defined as:

$$\dim_{\text{loc}} \mu(X) = \sup_{x \in X} \liminf_{r \to \infty} \frac{-\log \mu([x \upharpoonright r])}{r},$$

where $\mu([x \upharpoonright r])$ is understood as $\mu(x \upharpoonright r)$ whenever μ is a semimeasure.

In the language of supergales, the map $\boldsymbol{\delta}(\sigma) := 2^{s \cdot \text{len}(\sigma)} \cdot \mathbf{M}(\sigma)$ is an *optimal* constructive s-supergale whenever $s \geq 0$ is left-c.e. It follows from our Theorem 1.8.3 that $\dim_{\text{loc}} \mathbf{M}$ matches J. Lutz's *constructive dimension* [31,32]. That is, for any $X \subseteq 2^\omega$,

$$\begin{aligned} \dim_{\text{loc}} \mathbf{M}(X) &= \inf \{s \geq 0 : \text{there exists a constructive } s\text{-supergale succeeding on } X\} \\ &=: \text{cdim}(X). \end{aligned}$$

In light of (1.5), we may consider cdim to be an effective version of Hausdorff dimension. We will cite some other asymptotic coincidences further justifying this view.

In [45], Mayordomo proved that Lutz's constructive dimension of a point x matches the *lower incompressibility ratio of prefix Kolmogorov complexity* of x:

$$\text{cdim}(X) = \underline{\kappa}(X) := \sup_{x \in X} \liminf_{r \to \infty} \frac{K(x \upharpoonright r)}{r} = \dim_{\text{loc}} \mathbf{m}(X),$$

where the last equality follows from the Coding Theorem 1.8.2. So $\mathbf{M}$ and $\mathbf{m}$ produce the same local dimension notion on 2^ω. Note that by definition, $\dim_{\text{loc}} \mathbf{M}$ is itself the *lower incompressibility ratio with respect to a priori complexity*:

$$\dim_{\text{loc}} \mathbf{M}(X) = \sup_{x \in X} \liminf_{r \to \infty} \frac{\text{KM}(x \upharpoonright r)}{r}.$$

One might also attempt to effectivize the definition of classical Hausdorff dimension directly. Suppose $\mathcal{U}$ is a Martin-Löf-s-test covering $X \subseteq 2^\omega$. Then, $\mathcal{U}$ acts as an effective witness to X being s-null. In analogy with the definition of Hausdorff dimension in (1.1), define the *effective Hausdorff dimension* of $X \in 2^\omega$ to be:

$$\text{effdim}(X) := \inf \{s \geq 0 : X \text{ is ML-}s\text{-null}\}.$$

Note that the same could be done for any other of the effective test notions defined above. We prefer to work with Martin-Löf-randomness because it admits universal tests. In particular, for each left-c.e. $s > 0$, there exists a *universal* ML-s-test $\mathcal{U}^s$. This means that if $\mathcal{U}$ is an ML-s-test covering some subset $X \subseteq 2^\omega$, then so too will $\mathcal{U}^s$ cover X. The existence of such a $\mathcal{U}^s$ implies the *pointwise stability* of effective Hausdorff dimension. That is, for any $X \subseteq 2^\omega$,

$$\operatorname{effdim}(X) = \sup_{x \in X} \operatorname{effdim}(\{x\}).$$

Martin-Löf-s-randomness relates to success by constructive s-supergales. That is, if there exists a constructive s-supergale succeeding on X, then X will be ML-t-null for all left-c.e. $t > s$. Conversely, if s is left-c.e. and X is ML-s-null, then there exists a constructive s-supergale succeeding on X. Therefore, $\operatorname{cdim}(X) = \operatorname{effdim}(X)$ on all sets $X \subseteq 2^\omega$.

In his doctoral dissertation, N. Lutz introduced the map $\boldsymbol{\kappa}(X) := 2^{-K(X)}$, an outer measure on 2^ω (originally, $\mathbb{R}^m$) which possesses many algorithmic optimality properties as explored in both [35, 36]. In particular, J. Lutz and N. Lutz showed that $\boldsymbol{\kappa}(X)$ is *locally optimal* among all the other strongly finite, lower-semicomputable outer measures on 2^ω, and that any locally optimal outer measure $\boldsymbol{\mu}$ has local dimension $\dim_{\text{loc}} \boldsymbol{\mu}$ matching the lower incompressibility ratio of prefix complexity [35].

We now summarize these standard asymptotic coincidences in the following theorem.

Theorem 1.9.2 (Effective Dimension)**.** *For any $X \subseteq 2^\omega$, we have*

$$\operatorname{effdim}(X) = \dim_{\text{loc}} \mathbf{M}(X) = \dim_{\text{loc}} \mathbf{m}(X) = \dim_{\text{loc}} \boldsymbol{\kappa}(X) = \operatorname{cdim}(X) = \underline{\kappa}(X).$$

We will use $\dim(\cdot)$ as a generic placeholder for any one of those listed in Theorem 1.9.2 over either Cantor space or Euclidean space. And $\dim(x)$ will stand for $\dim(\{x\})$ for any $x \in 2^\omega$.

We further note that *effectivized packing dimension* (or *strong dimension*) coincides with the *upper* incompressibility ratio of prefix complexity, as first shown in [2]. So, we fix the notation:

$$\operatorname{Dim}(X) := \overline{\kappa}(X) = \sup_{x \in X} \limsup_{r \to \infty} \frac{K(x \restriction r)}{r}.$$

1.10 Effective Dimension Variants

We review two variants of effective dimension.

1.10.1 Mutual Dimension

One variant of effective dimension is *mutual dimension*, which was originally defined in [6]. It intuitively captures the algorithmic information shared by two infinitary objects in the limit. In Shannon information theory, mutual information is a measure of the correlation between two distributions. A. Case and J. Lutz's algorithmic notion of mutual information serves as a refinement of the classical notion. Namely, the *(conditional) mutual information between two finitary objects $a, b \in \mathscr{N}$ given $c \in \mathscr{N}$* is

$$I(a : b \mid c) := K(a \mid c) - K(a \mid b, c).$$

Extending their notation slightly, we define the *mutual information between subsets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$* as:

$$I(X : Y) := \min \{I(p : q) : p \in X \cap \mathbb{Q}^m \text{ and } q \in Y \cap \mathbb{Q}^n\},$$

or zero if taken over the empty set. Then the *mutual information between $x \in \mathbb{R}^m$ to precision-level r and $y \in \mathbb{R}^n$ to precision-level s* is defined as:

$$I_{r:s}(x : y) := I(B_{2^{-r}}(x) : B_{2^{-s}}(x)),$$

Both I and $I_{r:s}$ inherit many of their expected properties such as a chain rule and symmetry from K and $K_{r|s}$. In particular, a slight generalization to Theorem 4.10 in [6] for distinct precision-levels proves the chain rule for $I_{r:s}$.

Theorem 1.10.1 (Chain Rule for Mutual Information, c.f. Theorem 4.10 of [6])**.** *For all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \omega$,*

$$\begin{aligned} I_{r:s}(x : y) &= K_r(x) + K_s(y) - K_{r,s}(x, y) \pm (o(r) + o(s)) \\ &= K_r(x) + K_{s|r}(y \mid x) \pm (o(r) + o(s)). \end{aligned}$$

Case and Lutz further defined the *lower and upper mutual dimensions of $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$* as follows:

$$\operatorname{mdim}(x : y) := \liminf_{r\to\infty} \frac{I_{r:r}(x : y)}{r}, \quad \text{and} \quad \operatorname{Mdim}(x : y) := \limsup_{r\to\infty} \frac{I_{r:r}(x : y)}{r}.$$

Both mutual dimensions possess a few special properties. For instance, if $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, then $0 \leq \operatorname{mdim}(x : y) \leq \min\{\operatorname{mdim}(x), \operatorname{mdim}(y)\}$. As well as symmetry:

mdim$(x : y) = $ mdim$(y : x)$. Analogous statements hold for Mdim.

1.10.2 Conditional Dimension

A conditional version of effective dimension was first defined in [34] using the lifted conditional prefix complexity $K_{r|s}$. The *lower and upper conditional dimensions of* $x \in \mathbb{R}^m$ *given* $y \in \mathbb{R}^n$ were defined as:

$$\dim(x \mid y) = \liminf_{r \to \infty} \frac{K_{r|r}(x \mid y)}{r}, \quad \text{and} \quad \mathrm{Dim}(x \mid y) = \limsup_{r \to \infty} \frac{K_{r|r}(x \mid y)}{r}.$$

Conditional dimension may equivalently be characterized in terms of semimeasures and supergales as in Theorem 1.9.2.

These conditional dimensions were shown to relate with the weak and strong effective dimensions as follows:

Theorem 1.10.2 (Chain Rule for Conditional Dimension; Corollary 13 of [34])**.**

$$\dim(x) + \dim(y \mid x) \le \dim(x, y) \le \dim(x) + \mathrm{Dim}(y \mid x) \le \mathrm{Dim}(x, y) \le \mathrm{Dim}(x) + \mathrm{Dim}(y \mid x).$$

1.10.3 Results on Mutual and Conditional Dimensions

Both mutual dimensions behave predictably under computable, uniformly-continuous maps. We cite here just one corollary of this robustness.

Theorem 1.10.3 (Preservation of Mutual Dimension; Corollary 8.3 of [6])**.** *If* $f : \mathbb{R}^m \to \mathbb{R}^k$ *and* $g : \mathbb{R}^n \to \mathbb{R}^\ell$ *are computable and bi-Lipschitz, then for all* $x \in \mathbb{R}^m$ *and* $\mathbb{R}^n$*,*

$$\mathrm{mdim}(f(x) : g(y)) = \mathrm{mdim}(x : y),$$
$$\mathrm{Mdim}(f(x) : g(y)) = \mathrm{Mdim}(x : y).$$

In his dissertation [52], J. Reimann argues why effective dimension is also preserved under bi-computable, (locally) bi-Lipschitz continuous maps. This fact was used by N. Lutz to confirm an analog of Marstrand's Projection Theorem for the effective dimension of points [37]. Results like these mimic the preservation of Hausdorff dimension under locally bi-Lipschitz continuous maps (e.g., Corollary 2.4 in [12]). We will expand on this later in Section 2.2.

Finally, we summarize the known relations between mutual, conditional, and effective dimensions.

Proposition 1.10.4. *For all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$,*

$$\dim(x \mid y) \leq \dim(x) - \operatorname{mdim}(x : y) \leq \operatorname{Dim}(x \mid y),$$
$$\dim(x \mid y) \leq \operatorname{Dim}(x) - \operatorname{Mdim}(x : y) \leq \operatorname{Dim}(x \mid y).$$

We note that two of these inequalities were already proved in Corollary 11 of [34]. Our contribution is to add the uppermost bound in the top line, and the lowermost bound in the bottom line. All of these inequalities follow by re-expressing mutual dimension using the chain rules for $I_{r:r}$ and $K_{r|r}$ as in Theorems 1.10.1 and 1.7.6, respectively, and then distributing the limits inferior or superior appropriately over those terms.

1.11 Complexity Characterizations of Algorithmic Randomness

1.11.1 Algorithmic Discrepancies

We fix some notation for the quantitative discrepancy between a finitary object's algorithmic entropy and its mass according to some premeasure.

Definition 1.11.1. Let ρ be a premeasure on 2^ω. Define for any $\sigma \in 2^{<\omega}$ its:

- *A priori discrepancy*: $\Delta \mathrm{M}_\rho(\sigma) := \mathrm{KM}(\sigma) + \log \rho([\sigma]) = \log \frac{\rho([\sigma])}{\mathbf{M}(\sigma)}$,
- *Prefix discrepancy*: $\Delta \mathrm{P}_\rho(\sigma) := K(\sigma) + \log \rho([\sigma]) =^* \log \frac{\rho([\sigma])}{\mathbf{m}(\sigma)}$.

When $\rho = \rho_s$, simply write $\Delta \mathrm{M}_s$ and $\Delta \mathrm{P}_s$.

1.11.2 Complexity Characterizations of Algorithmic Randomness

It turns out that these algorithmic discrepancies characterize the various s-dimensional randomness notions defined above.

Theorem 1.11.2. *Let $s \geq 0$ be left-c.e. and $x \in 2^\omega$. Then x is:*

- *ML-s-random $\iff$ x is weak Chaitin-s-random: $\Delta \mathrm{P}_s(x \upharpoonright r) \geq O(1)$, [68];*
- *Solovay-s-random $\iff$ x is strong Chaitin-s-random: $\Delta \mathrm{P}_s(x \upharpoonright r) \to \infty$, [5, 68];*
- *Strong ML-s-random $\iff$ $\Delta \mathrm{M}_s(x \upharpoonright r) \geq O(1)$, [5];*

- *Vehement ML-s-random* $\iff$ *x is strong ML-s-random;*
- *Strong Solovay-s-random* $\iff$ $\Delta \mathrm{M}_s(x \upharpoonright r) \to \infty$, *[48].*

In light of these characterizations, one may argue on the level of discrete or continuous semimeasures rather than covers in order to effectivize results on Hausdorff dimension.

1.12 Point-to-Set Principles

In this context, a point-to-set (PTS) principle will characterize the fractal dimension or measure content of a subset by some local dimension or measure assigned to its elements. We summarize the few PTS principles known for Hausdorff dimension and the s-dimensional Hausdorff outer measures via algorithmically-restricted quantities.

1.12.1 PTS for Hausdorff Dimension

Certain fractal dimension notions have pointwise, algorithmic characterizations (over simple spaces such as Cantor space and Euclidean space). This was originally shown for Hausdorff dimension in 2017 by J. Lutz and N. Lutz [34].

Theorem 1.12.1 (Point-to-Set Principle; Theorem 1 of [34])**.** *For any* $X \subseteq 2^\omega$ *or* $X \subseteq \mathbb{R}^m$*, we have*

$$\dim_{\mathrm{H}} X = \min_{B \in 2^{\leq\omega}} \sup_{x \in X} \dim^B(x).$$

Notably, this implies that for each subset X, there exists an oracle $B \in 2^{\leq\omega}$ which is powerful enough to simulate Hausdorff dimension by ML-B-s-tests. Any such oracle is called a *Hausdorff oracle* for X. An analogous formula holds for classical packing dimension (see Theorem 2 of [34]), swapping weak effective dimension dim for strong dimension: Dim.

The main insight used to prove the Point-to-Set Principle involves the ability to build for a fixed set $X \subseteq \mathbb{R}^m$ an oracle B with knowledge of how X intersects with any given dyadic cube from $\mathcal{Q}^m$. Arbitrary covers of X exhibiting $\dim_{\mathrm{H}} X$ may be approximated using covers comprising dyadic cubes, and so the Hausdorff dimension of X is the supremum of the size of the intersection with X along nested sequences of dyadic cubes. Suppose that the intersection of a nested sequence intersecting with X is the singleton set containing $x \in X$. Then $\dim^B(x)$ exactly captures how much dimension X takes on at x.

In practice, the power of the Point-to-Set Principle is in proving lower-bounds for Hausdorff dimension. As remarked to me by Ryan Bushling, one might as well consider this a *local-to-global* principle for the following reason: in order to witness a lower bound on $\dim_{\mathrm{H}} X$, it is not enough to find a single sufficiently-random element $x \in X$, for any countable number of reals could be coded into an oracle. Instead, one must demonstrate the existence of *many* highly-complex elements of X, which requires knowledge of a large portion of the set to do.

The Point-to-Set Principle has found applications across geometric measure theory, including in bounding the Hausdorff dimension of Furstenberg sets [40, 65], lineal extensions [4], distance sets [1], and pinned-distance sets [67]; as well as in extending the Marstrand-Mattila Projection Theorem [66].

Well before the full Point-to-Set Principle was proved, L. Staiger [63] and B. Ryabko [56, 57] had already observed versions of the Point-to-Set Principle for simply-definable classes such as Σ^0_2-classes. This was slightly generalized by J. Hitchcock in [17] in what he calls a *correspondence principle* between constructive dimension and Hausdorff dimension [17]. The spirit of any correspondence principle is to characterize a classical notion by unrelativized, effective means (i.e., avoiding oracles).

Theorem 1.12.2 (Correspondence Principle for Hausdorff dimension, Theorem 5.3 of [17])**.** *For any Σ^0_2-class $X \subseteq 2^\omega$, we have*

$$\dim_{\mathrm{H}} X = \dim(X).$$

Hitchcock further showed that this correspondence broke down immediately beyond this level of the lightface Borel hierarchy.

1.12.2 PTS for Hausdorff Measures

The original Point-to-Set Principle has recently been refined to the s-dimensional Hausdorff outer measures by P. Lutz and J. Miller [41]. These results also expand on the complexity characterizations of various partial randomness notions from Theorem 1.11.2.

Theorem 1.12.3. *For every $X \subseteq 2^\omega$ and $s \geq 0$,*

$$\log \mathcal{H}^s(X) =^+ \inf_{B \in 2^{\leq\omega}} \sup_{x \in X} \liminf_{r \to \infty} \Delta\mathrm{M}^B_s(x \upharpoonright r),$$

and X is not σ-finite with respect to $\mathcal{H}^s$ if and only if

$$(\forall B \in 2^{\leq\omega})(\exists x \in X)\left[\liminf_{r\to\infty} \Delta\mathrm{M}_s^B(x \upharpoonright r) = \infty\right].$$

By Theorem 1.11.2, X is not σ-finite in dimension s if and only if there is a strong Solovay-B-s-random element in X for every oracle B.

Corollary 1.12.4. *For every $X \subseteq 2^\omega$ and $s \geq 0$,*

$$\begin{aligned}
\mathcal{H}^s(X) > 0 &\iff (\forall B \in 2^{\leq\omega})(\exists x \in X)\left[\liminf_{r\to\infty} \Delta\mathrm{M}_s^B(x \upharpoonright r) > -\infty\right] \\
&\iff (\forall B \in 2^{\leq\omega})(\exists x \in X)\left[\liminf_{r\to\infty} \Delta\mathrm{P}_s^B(x \upharpoonright r) > -\infty\right], \\
\mathcal{H}^s(X) < \infty &\iff (\exists B \in 2^{\leq\omega})\left[\sup_{x\in X}\liminf_{r\to\infty} \Delta\mathrm{M}_s^B(x \upharpoonright r) < \infty\right].
\end{aligned}$$

Again by Theorem 1.11.2, X is not s-null if and only if there is an ML-B-s-random element in X for every oracle B.

In effect, these refined point-to-set principles demonstrate the usefulness of *a priori* complexity–defined by continuous semimeasures–in characterizing not just Hausdorff dimension but also the s-dimensional Hausdorff outer measures.

Chapter 2 Robustness of Conditional Kolmogorov Complexity

2.1 Robustness of the Lift

2.1.1 An Alternate Lift

Recall that in Section 1.7, we specified the standard lift of conditional prefix complexity to Euclidean space. We claim that the order of maximum and minimum in its definition makes no difference asymptotically. That is, suppose we were to alternatively define for $B \subseteq \mathbb{R}^m$ and $C \subseteq \mathbb{R}^n$:

$$\mathcal{K}(B \mid C) := \min_p \left\{ \max_q \left\{ K(p \mid q) : q \in C \cap \mathbb{Q}^n \right\} : p \in B \cap \mathbb{Q}^m \right\},$$

and for $x \in \mathbb{R}^m$, $q \in \mathbb{Q}^m$, $y \in \mathbb{R}^n$, and $r, s \in \omega$:

$$\begin{aligned} \mathcal{K}_{r|s}(x \mid y) &:= \mathcal{K}(B_{2^{-r}}(x) \mid B_{2^{-s}}(y)), \text{ and} \\ \mathcal{K}_s(p \mid y) &:= \mathcal{K}(\{p\} \mid B_{2^{-s}}(y)). \end{aligned}$$

Theorem 2.1.1 (Robustness of Euclidean Conditional Kolmogorov Complexity)**.** *Let $m, n \in \omega$. Then, for any $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \omega$,*

$$\mathcal{K}_{r|s}(x \mid y) = K_{r|s}(x \mid y),$$

where the equality holds up to $2K(r) + 2K(s) + O_{m,n}(1)$.

We will obtain the above theorem by applying the same proof methods found in [6] on the linear sensitivities of K in its precision parameters. In particular, both forms K and $\mathcal{K}$ are well-approximated by dyadic truncation as in Lemma 1.7.5. We first recall an important geometric fact from [6]. In Euclidean space $\mathbb{R}^m$, it takes a sufficiently small scaling factor $\gamma > 0$ in order to guarantee at least one node in the lattice $\gamma\mathbb{Z}^m$ lies in a given ball of fixed radius.

Observation 2.1.2 (Observation 3.4 in [6], Observation 3 in [34])**.** For every $m \in \omega$, $r > 0$, and open ball $B \subseteq \mathbb{R}^m$ of radius 2^{-r},

$$B \cap 2^{-\left(r+\lceil \log \sqrt{m} \rceil\right)}\mathbb{Z}^m \neq \emptyset.$$

Now, we prove linear sensitivities for the alternate lift $\mathcal{K}$.

Lemma 2.1.3 (Linear Sensitivity in the Condition for $\mathcal{K}$)**.** *For all $m, n \in \omega$, $p \in \mathbb{Q}^m$, $y \in \mathbb{R}^n$, and $s, \Delta s \in \omega$,*

$$\mathcal{K}_{s+\Delta s}\left(p \mid y\right) \leq \mathcal{K}_{s}\left(p \mid y\right) \leq \mathcal{K}_{s+\Delta s}\left(p \mid y\right) + n\Delta s + K(s) + O_n(\log \Delta s) + O_n(1).$$

Proof. The first inequality follows by definition. Unpacking the notation, it suffices to show that for any fixed $q \in B_{2^{-s}}(y) \cap \mathbb{Q}^n$,

$$K\left(p \mid q\right) \leq \mathcal{K}_{s+\Delta s}\left(p \mid y\right) + n\Delta s + K(s) + O_n(\log \Delta s) + O_n(1).$$

Note that we have the ability to compute a rational element of the smaller open ball $B_{2^{-(s+\Delta s)}}(y)$ from the fixed, rational element q of the larger open ball $B_{2^{-s}}(y)$: by Observation 2.1.2, there exists an integer tuple $z \in \mathbb{Z}^n$ such that $2^{-s^*}z \in B_{2^{-(s+\Delta s)}}(y - q)$, where $s^* = s + \Delta s + \lceil \log \sqrt{n} \rceil$. And set $q' := 2^{-s^*}z + q \in B_{2^{-(s+\Delta s)}}(y)$. Notice that

$$||z|| = 2^{s^*}\left|\left|2^{-s^*}z\right|\right| \leq 2^{s^*}\left(2^{-s} + 2^{-(s+\Delta s)}\right) = 2^{\lceil \log \sqrt{n} \rceil}\left(2^{\Delta s} + 1\right) \leq 2^{\Delta s + \lceil \log \sqrt{n} \rceil + 1}.$$

By (1.4), we have

$$K(z) \leq n\Delta s + O_n(\log \Delta s) + O_n(1).$$

So, we may conclude

$$\begin{aligned} K\left(p \mid q\right) &\leq K\left(p \mid q'\right) + K\left(q' \mid q\right) + O_n(1) \\ &\leq \mathcal{K}_{s+\Delta s}\left(p \mid y\right) + K\left(z\right) + K\left(s, \Delta s\right) + O_n(1) \end{aligned}$$

$$\leq \mathcal{K}_{s+\Delta s}(p \mid y) + n\Delta s + K(s) + O_n(\log \Delta s) + O_n(1).$$

□

Lemma 2.1.4 (Linear Sensitivity in the Input for $\mathcal{K}$)**.** *For all $m, n \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, \Delta r, s \in \omega$,*

$$\mathcal{K}_{r|s}(x \mid y) \leq \mathcal{K}_{r+\Delta r|s}(x \mid y) \leq \mathcal{K}_{r|s}(x \mid y) + m\Delta r + K(r) + O_m(\log \Delta r) + O_{m,n}(1).$$

Proof. The first inequality follows by definition. Unpacking the notation, it suffices to show for any fixed $p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m$ that

$$\mathcal{K}_{r+\Delta r|s}(x \mid y) \leq \mathcal{K}_s(p \mid y) + m\Delta r + K(r) + O_m(\log \Delta r) + O_{m,n}(1).$$

Note that we have the ability to compute some element of the smaller open ball $B_{2^{-(r+\Delta r)}}(x)$ from a fixed, rational element p of the larger open ball $B_{2^{-r}}(x)$: by Observation 2.1.2, there exists an integer tuple $z \in \mathbb{Z}^m$ for which $2^{-r^*}z \in B_{2^{-(r+\Delta r)}}(x-p)$, where $r^* = r + \Delta r + \lceil \log \sqrt{m} \rceil$. Then, we take $p' := p + 2^{-r^*}z \in B_{2^{-(r+\Delta r)}}(x)$. Thus, by (1.4),

$$\begin{aligned}
\mathcal{K}_{r+\Delta r|s}(x \mid y) &\leq \mathcal{K}_s(p' \mid y) \\
&\leq \mathcal{K}_s(p \mid y) + K(p' \mid p) + O_{m,n}(1) \\
&\leq \mathcal{K}_s(p \mid y) + K(z) + K(r, \Delta r) + O_{m,n}(1) \\
&\leq \mathcal{K}_s(p \mid y) + m\Delta r + K(r) + O_m(\log \Delta r) + O_{m,n}(1).
\end{aligned}$$

□

Lemma 2.1.5 (Dyadic Truncation Suffices for Condition in $\mathcal{K}$)**.** *For all $m, n \in \omega$, $p \in \mathbb{Q}^m$, $y \in \mathbb{R}^n$, and $s \in \omega$,*

$$|\mathcal{K}_s(p \mid y) - K(p \mid y \upharpoonright s)| \leq K(s) + O_n(1).$$

Proof. In one direction, we have that $y \upharpoonright s$ is a rational in a ball about y of radius $2^{-s}\sqrt{n}$. So, $y \upharpoonright s$ is considered in the evaluation of $\mathcal{K}_{s+\log\sqrt{n}}(p \mid y)$. We appeal to linear sensitivity 2.1.3 in s and use the fact that $y \upharpoonright s \in B_{2^{-s}\sqrt{n}}(y) \cap \mathbb{Q}^n$ to bound $\mathcal{K}_s(p \mid y)$ from below:

$$\begin{aligned}
\mathcal{K}_s(p \mid y) &\geq \mathcal{K}_{s-\log\sqrt{n}}(p \mid y) - n\log\sqrt{n} - K(s) - O(\log\log\sqrt{n}) - O(1) \\
&\geq K(p \mid y \upharpoonright s) - K(s) - O(n \log n) - O(1).
\end{aligned}$$

In the other direction, unpacking the notation, it suffices to check that:

$$K(p \mid q) \leq K(p \mid y \upharpoonright s) + K(s) + O_n(1)$$

for any fixed $q \in B_{2^{-s}}(y) \cap \mathbb{Q}^n$. Because $y \upharpoonright s \in B_{2^{-s}(1+\sqrt{n})}(q) \cap \mathcal{Q}^n_s$ is contained in a set of cardinality no greater than the constant $(2(1+\sqrt{n}))^n$, we may describe $y \upharpoonright s$ using q, s, and a constant number of extra bits depending on n. So,

$$\begin{aligned} K(p \mid q) &\leq K(p \mid y \upharpoonright s) + K(y \upharpoonright s \mid q) + K(s) + O_n(1) \\ &\leq K(p \mid y \upharpoonright s) + K(s) + O_n(1). \end{aligned}$$

□

Lemma 2.1.6 (Dyadic Truncation Suffices for both Parameters in $\mathcal{K}$)**.** *For all $m, n \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \omega$,*

$$\left|\mathcal{K}_{r|s}(x \mid y) - K(x \upharpoonright r \mid y \upharpoonright s)\right| \leq K(r) + K(s) + O_{m,n}(1).$$

We could prove this lemma directly using the linear sensitivity of $\mathcal{K}$ in r in Lemma 2.1.4. But there is a simpler proof using dyadic truncation.

Proof. Dropping the logarithmic terms in the precision-levels and norms, and using the definitions of K and $\mathcal{K}$:

$$\begin{aligned} \mathcal{K}_{r|s}(x \mid y) &:= \min_p \{\mathcal{K}_s(p \mid y) : p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} \\ &= \min_p \{K(p \mid y \upharpoonright s) : p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} && \text{[Lemma 2.1.5]} \\ &= K_r(x \mid y \upharpoonright s) \\ &= K(x \upharpoonright r \mid y \upharpoonright s). && \text{[Lemma 1.7.4]} \end{aligned}$$

□

Together, the dyadic truncation lemmas 1.7.5 and 2.1.6 imply that $K(x \upharpoonright r \mid y \upharpoonright s)$ approximates both $K_{r|s}(x \mid y)$ and $\mathcal{K}_{r|s}(x \mid y)$ up to logarithmic terms in r and s, proving Theorem 2.1.1.

2.1.2 Approximation by K-Minimizers

Conditional complexity is not only well-approximated by dyadic truncation, but also by K-minimizers. In particular, we prove conditional versions of the result from [6] stating that the lift of conditional complexity to Euclidean space can also be well-approximated by evaluating the conditional complexity of the corresponding K-minimizers. This will imply a conditionalized chain rule for conditional complexity.

Our method makes use of some basic facts established in [6] intuitively saying that any K-minimizer of an open ball represents the information content essentially possessed by all the other rational points in the ball.

Lemma 2.1.7 (Lemma 4.3 of [6])**.** *Let $m, n, \ell \in \omega$, $p \in \mathbb{Q}^m$, $w \in \mathbb{Q}^\ell$, $y \in \mathbb{R}^n$, and $s \in \omega$. If $q \in B_{2^{-s}}(y) \cap \mathbb{Q}^n$ and q^* is a K-minimizer of $B_{2^{-s}}(y)$, then*

$$K(p \mid q, w) \leq K(p \mid q^*, w) + K(K(q^*)) + O_n(\log s) + O_n(1).$$

Corollary 2.1.8 (Corollary 4.4 in [6])**.** *Let $m \in \omega$ and $x \in \mathbb{R}^m$. Suppose $p^* \in \mathbb{Q}^m$ is a K-minimizer of $B_{2^{-r}}(x)$, and $p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m$ is some other nearby rational. Then*

$$K(p^* \mid p) = O_m(\log r) + O_m(1).$$

Now, we may conclude that K-minimizers suffice for computing conditional complexity. We use here that the two lifts of prefix complexity asymptotically agree: $K \approx \mathcal{K}$, where the approximate equality $\approx$ holds up to sub-linear terms in the precision-levels. For the rest of the section, we write (in)equalities up to the logarithmic terms:

$$O_m(\log r) + O_n(\log s) + O_{m,n}(1). \tag{2.1}$$

Proposition 2.1.9 (K-minimizers Suffice for Conditional Complexity)**.** *Let $m, n \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \omega$. If $p^* \in \mathbb{Q}^m$ and $q^* \in \mathbb{Q}^n$ are each K-minimizers of $B_{2^{-r}}(x)$ and $B_{2^{-s}}(y)$, respectively, then*

$$K_{r|s}(x \mid y) = K(p^* \mid q^*),$$

where equality holds up to the logarithmic factors given by (2.1)*.*

Proof. Let (p^*, q^*) be the K-minimizers for x and y at precisions r and s, respectively. Additionally, let (p', q') denote a witness pair to $K_{r|s}(x \mid y)$, and (p'', q'') the same for

$\mathcal{K}_{r|s}(x \mid y)$. Then, dropping any logarithmic terms,

$$\begin{aligned}
K_{r|s}(x \mid y) &= \mathcal{K}_{r|s}(x \mid y) && \text{[Theorem 2.1.1]} \\
&= K(p'' \mid q'') \\
&\leq K(p^* \mid q'') \\
&= K(p^* \mid q^*) + K(q^* \mid q'') \\
&= K(p^* \mid q^*) && \text{[Corollary 2.1.8]} \\
&= K(p^* \mid p') + K(p' \mid q^*) \\
&\leq K(p' \mid q^*) && \text{[Corollary 2.1.8]} \\
&\leq K(p' \mid q') \\
&= K_{r|s}(x \mid y).
\end{aligned}$$

□

Now, we aim to prove an approximate, conditional chain rule for $K_{r,s|t}$. Theorem 1.10.1 captures the approximate, unconditional chain rule for $K_{r|s}$, and it implicitly follows by applying the arguments in Section 4 of [6] under the distinct precision-levels r and s. We may achieve the conditional version using approximations by K-minimizers.

Lemma 2.1.10. *Let $m, n, \ell \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^\ell$, and $r, s, t \in \omega$. Denote by p^*, q^*, w^* the K-minimizers of $B_{2^{-r}}(x)$, $B_{2^{-s}}(y)$, and $B_{2^{-t}}(z)$, respectively. Then the following equalities hold up to the logarithmic factors given by* (2.1)*:*

(i) $K_{r,s|t}(x, y \mid z) = K(p^*, q^* \mid w^*)$,

(ii) $K_{r|s,t}(x, y \mid z) = K(p^* \mid q^*, w^*)$.

Proof. We have two competing notions of K-minimizer, now: either unconditional, or conditioned on some extra information $\zeta \in 2^{<\omega}$. Let $p^* \in \mathbb{Q}^m$ be a K-minimizer of $B_{2^{-r}}(x)$ and $p^*_\zeta \in \mathbb{Q}^m$ a witness to $K_r(x \mid \zeta)$. We first show that, no matter ζ, the K-minimizer p^* also approximately minimizes $K_r(x \mid \zeta)$. By definition,

$$\begin{aligned}
K(p^*_\zeta \mid \zeta) &\leq K(p^* \mid \zeta) \\
&\leq K(p^*_\zeta \mid \zeta) + K(p^* \mid p^*_\zeta) \\
&\leq K(p^*_\zeta \mid \zeta) && \text{[Lemma 2.1.7]} \\
&= K_r(x \mid \zeta). && (2.2)
\end{aligned}$$

This implies a conditional version of Corollary 2.1.8, where for any $p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m$,

$$\begin{aligned}
K(p^*_\zeta \mid p, \zeta) &\leq K(p^*_\zeta \mid p^*, \zeta) && \text{[Lemma 2.1.7]} \\
&= K(p^*_\zeta \mid \zeta) + K(p^* \mid p^*_\zeta, \zeta) - K(p^* \mid \zeta) \\
&= K(p^* \mid p^*_\zeta, \zeta) && \text{[(2.2)]} \\
&= 0. && \text{[Corollary 2.1.8]} \qquad (2.3)
\end{aligned}$$

Now, we assert $K_{r,s|t}(x, y \mid z) \approx K(p^*, q^* \mid w^*)$. Call w' a maximizing witness to z in the definition of $K_{r,s|t}(x, y \mid z)$, and let $(p^*_{w'}, q^*_{w'})$ be minimizing witnesses of $K_r(x \mid w')$ and $K_s(y \mid w')$, respectively. We follow very similar steps to the proof of Proposition 2.1.9. In one direction:

$$\begin{aligned}
K_{r,s|t}(x, y \mid z) = K_{r,s}(x, y \mid w') &= K(p^*_{w'}, q^*_{w'} \mid w') && \text{[Proposition 2.1.9]} \\
&= K(p^*_{w'} \mid w') + K(q^*_{w'} \mid p^*_{w'}, w') \\
&= K(p^* \mid w') + K(q^*_{w'} \mid p^*_{w'}, w') && \text{[(2.2)]} \\
&\leq K(p^* \mid w') + K(q^*_{w'} \mid p^*, w') && \text{[Lemma 2.1.7]} \\
&= K(q^*_{w'} \mid w') + K(p^* \mid q^*_{w'}, w') \\
&\leq K(q^* \mid w') + K(p^* \mid q^*, w') && \text{[Lemma 2.1.7]} \\
&= K(p^*, q^* \mid w') \\
&\leq K(p^*, q^* \mid w^*). && \text{[Lemma 2.1.7]}
\end{aligned}$$

In the reverse direction:

$$\begin{aligned}
K(p^*, q^* \mid w^*) &\leq K(p^*, q^* \mid w') \\
&= K(p^* \mid w') + K(q^* \mid p^*, w') \\
&\leq K(p^*_{w'} \mid w') + K(q^* \mid p^*_{w'}, w') + K(p^*_{w'} \mid p^*, w') && \text{[(2.2)]} \\
&= K(q^* \mid w') + K(p^*_{w'} \mid q^*, w') && \text{[(2.3)]} \\
&\leq K(q^*_{w'} \mid w') + K(p^*_{w'} \mid q^*_{w'}, w') + K(q^*_{w'} \mid q^*, w') && \text{[(2.2)]} \\
&= K(p^*_{w'}, q^*_{w'} \mid w') && \text{[(2.3)]} \\
&= K_{r,s|t}(x, y \mid z).
\end{aligned}$$

Finally, we move on to proving $K_{r|s,t}(x \mid y, z) \approx K(p^* \mid q^*, w^*)$. Let (q', w') be a pair of rationals witnesses for y and z in evaluating $K_{r|s,t}(x \mid y, z)$, and let $p^*_{q',w'} \in B_{2^{-r}}(x) \cap \mathbb{Q}^m$

minimize $K_r(x \mid q', w')$. Then,

$$
\begin{aligned}
K_{r|s,t}(x \mid y, z) \leq K(p^* \mid q', w') &\leq K(p^* \mid q^*, w^*) && \text{[Lemma 2.1.7]} \\
&\leq K(p^* \mid q', w') \\
&= K(p^*_{q',w'} \mid q', w') && \text{[(2.2)]} \\
&\leq K\left(p' \mid q', w'\right) = K_{r|s,t}(x \mid y, z).
\end{aligned}
$$

□

Note that Lemma 1.7.5 stated with extra conditional information follows from an identical proof to the unconditional version. So, corresponding linear sensitivities hold for $K_{r,s|t}(x, y \mid z)$; and dyadic truncation also approximates prefix complexity well in this more general case: $K_{r,s|t}(x, y \mid z) \approx K(x \upharpoonright r, y \upharpoonright s \mid z \upharpoonright t)$.

The other prefix complexity term from the statement of Lemma 2.1.10 is likewise well-approximated through dyadic truncations:

$$
\begin{aligned}
K_{r|s,t}(x \mid y, z) &= K(p^* \mid q^*, w^*) - K(q^* \mid w^*) \\
&= K(x \upharpoonright r, y \upharpoonright s \mid z \upharpoonright t) - K(y \upharpoonright s \mid z \upharpoonright t) && \text{[Lemma 1.7.5]} \\
&= K(x \upharpoonright r \mid y \upharpoonright s, z \upharpoonright t).
\end{aligned}
$$

Finally, Lemma 2.1.10 yields the fully conditional chain rule for conditional prefix complexity over Euclidean space.

Theorem 2.1.11 (Conditional Chain Rule for Conditional Complexity)**.** *Let $m, n, \ell \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^\ell$, and $r, s, t \in \omega$, then the following equalities hold up to the logarithmic factors given by* (2.1)*:*

$$K_{r,s|t}(x, y \mid z) = K_{r|t}(x \mid z) + K_{s|r,t}(y \mid x, z).$$

Proof. We simply follow the approximate equalities from the previous result:

$$
\begin{aligned}
K_{r,s|t}(x, y \mid z) &= K(p^*, q^* \mid w^*) && \text{[Lemma 2.1.10]} \\
&= K(p^* \mid w^*) + K(q^* \mid p^*, w^*) \\
&= K_{r|t}(x \mid z) + K_{s|r,t}(y \mid x, z). && \text{[Lemma 2.1.10]}
\end{aligned}
$$

□

2.2 Effective Dimension under Uniformly Continuous Maps

2.2.1 Uniform and Lipschitz Continuities

A *uniformly continuous function* is a real function which has a modulus of uniform continuity.

Definition 2.2.1. A *modulus of uniform continuity* for the function $f : \mathbb{R}^m \to \mathbb{R}^n$ is any non-decreasing function $\mu : \omega \to \omega$ such that, for all $x_1, x_2 \in \mathbb{R}^m$ and $r \in \omega$,

$$||x_1 - x_2|| \leq 2^{-\mu(r)} \implies ||f(x_1) - f(x_2)|| \leq 2^{-r}.$$

In particular, a uniformly continuous function is called *Hölder continuous* if it has a modulus of uniform continuity of the form $\mu(r) = \lceil (r + c)/\alpha \rceil$ where $\alpha > 0$ and $c \in \mathbb{R}$; and *Lipschitz continuous* if it has a modulus of uniform continuity of the form $\mu(r) = r + O(1)$.

An injection $f : \mathbb{R}^m \to \mathbb{R}^n$ is called *co-uniformly continuous* if it has a uniformly continuous inverse map. We use the prefix "co" similarly for maps with Hölder continuous or Lipschitz continuous inverses. Any modulus of uniform continuity for the inverse of a co-uniformly continuous map f is called an *inverse-modulus* for f. A bijection $f : \mathbb{R}^m \to \mathbb{R}^m$ is called *bi-uniformly continuous* if f is both uniformly continuous and co-uniformly continuous. We similarly use "bi" in the case of a map and its inverse both being Hölder or Lipschitz continuous.

Clearly, a map $f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous if and only if there exists $L \geq 0$ such that for each $x_1, x_2 \in \mathbb{R}^m$,

$$||f(x_1) - f(x_2)|| \leq L \cdot ||x_1 - x_2|| \, .$$

We refer to L as a *Lipschitz constant* for f.

Likewise, an injection $f : \mathbb{R}^m \to \mathbb{R}^n$ is co-Lipschitz continuous if and only if there exists $C \geq 0$ such that for each $x_1, x_2 \in \mathbb{R}^m$,

$$C \cdot ||x_1 - x_2|| \leq ||f(x_1) - f(x_2)|| \, .$$

We refer to C as a *co-Lipschitz constant* for f.

More generally, a map $f : \mathbb{R}^m \to \mathbb{R}^n$ is Hölder continuous if and only if there exist constants $\alpha > 0$ and $L > 0$ such that for each $x_1, x_2 \in \mathbb{R}^m$:

$$||f(x_1) - f(x_2)|| \leq L \cdot ||x_1 - x_2||^{\alpha}.$$

We refer to α as an *exponent* for f.

There is also a notion of *local Lipschitz continuity*. Call a function $f : \mathbb{R}^m \to \mathbb{R}^n$ defined on open set to be *locally Lipschitz continuous* if for every $x \in \text{dom}(f)$, there exist an open neighborhood $U \subseteq \text{dom}(f)$ containing x and a constant $L > 0$ such that for each $x_1, x_2 \in U$, we have

$$||f(x_1) - f(x_2)|| \leq L \cdot ||x_1 - x_2||.$$

Analogous local definitions exist for the other uniform continuity notions.

2.2.2 Known Results

Hausdorff dimension is invariant under bi-Lipschitz continuous transformations. This follows from the classical result about Hausdorff dimension under Hölder continuous transformations (see Proposition 2.2 of [12]). Yet, there is also an effective proof due to J. Reimann [52]. Reimann further effectivized the result to get a similar statement for effective dimension. In particular, let f be a computable, Hölder continuous transformation on 2^ω with exponent $\alpha > 0$. Then f is *α-expansive*, meaning for any $x \in 2^\omega$, we have $\text{len}(f(x) \upharpoonright r) \geq \alpha \cdot r$ for all but finitely many $r \in \omega$. This gives for each $x \in 2^\omega$ and $r \in \omega$:

$$K(f(x) \upharpoonright (\alpha \cdot r)) \leq^+ K(f(x \upharpoonright r)) \leq^+ K(x \upharpoonright r).$$

So, for any $X \subseteq 2^\omega$, we have $\dim(f(X)) \leq \dim(X)/\alpha$ in the limit (see Theorem 2.27 of [52]). Thus, effective dimension is invariant under bi-computable, bi-Lipschitz continuous maps. This fact extends to bi-computable maps which are only *locally* bi-Lipschitz continuous.

These basic results mimic the data processing inequalities for mutual dimension by A. Case and J. Lutz [6]. For instance, recall that Theorem 1.10.3 states the mutual dimension between two reals is invariant under computable, bi-Lipschitz maps. Here, we establish analogous results for conditional dimension.

2.2.3 Modulus and Data Processing for Conditional Dimension

We begin our analysis with an analogue of Lemma 6.3 from [6]–which applies to the lift of mutual information to Euclidean space–to clarify how conditional prefix complexity behaves under computable, uniformly continuous functions. If f is a computable, Lipschitz continuous map, then there is a computable real number $L \in \mathbb{Q}^+$ serving as a Lipschitz constant for f. That is, f has computable modulus $\mu(r) := r + \log L$. With this in mind, we may now state the *modulus and data processing inequalities* for conditional prefix Kolmogorov complexity lifted to Euclidean space.

Recall that we write any (in)equalities for Kolmogorov complexity for the rest of the section up to the sub-linear factors in the precision-levels given by (2.1). In contrast, the corresponding (in)equalities for effective dimension are exact.

Lemma 2.2.2 (Forward Modulus and Data Processing Inequalities for Conditional Complexity). *Let $m, n, k \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \omega$.*

(i) If $f : \mathbb{R}^m \to \mathbb{R}^k$ is computable and uniformly continuous with computable modulus μ, then

$$K_{r|s}(f(x) \mid y) \leq K_{\mu(r+1)|s}(x \mid y).$$

In particular, if f is also Lipschitz continuous,

$$\dim(f(x) \mid y) \leq \dim(x \mid y).$$

(ii) If $f : \mathbb{R}^n \to \mathbb{R}^k$ is computable and uniformly continuous with computable modulus μ, then

$$K_{r|\mu(s+1)}\left(x \mid y\right) \leq K_{r|s}\left(x \mid f(y)\right).$$

In particular, if f is also Lipschitz continuous,

$$\dim(x \mid y) \leq \dim(x \mid f(y)).$$

Proof. We start by proving (i). Let $p^*_{f(x)} \in B_{2^{-r}}(f(x))$, $p^*_x \in B_{2^{-\mu(r+1)}}(x)$ and $q^* \in B_{2^{-s}}(y)$ be K-minimizers in their respective open balls. Since f is computable, there is an oracle prefix-free Turing machine M which approximates f arbitrarily well at p^*_x when given p^*_x as an oracle; say, such that for any $r \in \omega$ we have $\left|M^{p^*_x}(r) - f(p^*_x)\right| \leq 2^{-(r+1)}$. Since μ is a modulus for f, M also approximates f arbitrarily well at x when given p_x^* as an oracle, since for any r,

$$\left|M^{p_x^*}(r) - f(x)\right| \leq \left|M^{p_x^*}(r) - f(p_x^*)\right| + |f(p_x^*) - f(x)| \leq 2^{-(r+1)} + 2^{-(r+1)} = 2^{-r}.$$

We use $M^{p_x^*}(r) \in B_{2^{-r}}(f(x))$ to conclude:

$$\begin{aligned}
K_{r|s}\left(f(x) \mid y\right) &= K\left(p_{f(x)}^* \mid q^*\right) && \text{[Proposition 2.1.9]}\\
&\leq K\left(p_{f(x)}^* \mid M^{p_x^*}(r)\right) + K\left(M^{p_x^*}(r) \mid q^*\right) \\
&\leq K\left(M^{p_x^*}(r) \mid q^*\right) && \text{[Corollary 2.1.8]}\\
&\leq K\left(p_x^* \mid q^*\right) && \text{[Theorem 1.6.2]}\\
&= K_{\mu(r+1)|s}\left(x \mid y\right). && \text{[Proposition 2.1.9]}
\end{aligned}$$

So, if f is indeed Lipschitz continuous, then there is a computable modulus for f of the form $\mu(r) = r + O(1)$, giving

$$\begin{aligned}
\dim(f(x) \mid y) &= \liminf_{r\to\infty} \frac{K_{r|r}(f(x) \mid y)}{r} \\
&\leq \liminf_{r\to\infty} \frac{K_{\mu(r+1)|r}(x \mid y)}{r} \\
&= \liminf_{r\to\infty} \frac{K_{r+O(1)|r}(x \mid y)}{r} \\
&= \liminf_{r\to\infty} \frac{K_{r|r}(x \mid y)}{r} && \text{[Lemma 1.7.3(i)]}\\
&= \dim(x \mid y).
\end{aligned}$$

The proof of (ii) is similar. This time, let $p^* \in B_{2^{-r}}(x)$, $q_y^* \in B_{2^{-\mu(s+1)}}(y)$, and $q_{f(y)}^* \in B_{2^{-s}}(f(y))$ be K-minimizers in their respective open balls. Since f is computable, there is an oracle prefix-free Turing machine M which approximates f arbitrarily well at q_y^* when given q_y^* as an oracle: $\left|M^{q_y^*}(s) - f(q_y^*)\right| \leq 2^{-(s+1)}$. Since μ is a modulus for f, we get that M also approximates f arbitrarily well at y when given q_x^* as an oracle:

$$\left|M^{q_y^*}(s) - f(y)\right| \leq \left|M^{q_y^*}(s) - f(q_s^*)\right| + |f(q_s^*) - f(y)| \leq 2^{-(s+1)} + 2^{-(s+1)} = 2^{-s}.$$

We use $M^{q_y^*}(s) \in B_{2^{-s}}(f(y))$ to conclude:

$$K_{r|\mu(s+1)}\left(x \mid y\right) = K\left(p^* \mid q_y^*\right) \qquad \text{[Proposition 2.1.9]}$$

$$
\begin{aligned}
&= K\left(p^* \mid M^{q_y^*}(s)\right) + K\left(M^{q_y^*}(s) \mid q_y^*\right) \\
&\leq K\left(p^* \mid M^{q_y^*}(s)\right) + K\left(q_y^* \mid q_y^*\right) && \text{[Theorem 1.6.2]} \\
&\leq K\left(p^* \mid q_{f(y)}^*\right) + K\left(q_{f(y)}^* \mid M^{q_y^*}(s)\right) \\
&= K\left(p^* \mid q_{f(y)}^*\right) && \text{[Corollary 2.1.8]} \\
&= K_{r|s}\left(x \mid f(y)\right). && \text{[Proposition 2.1.9]}
\end{aligned}
$$

So, if f is indeed Lipschitz continuous, then there is a computable modulus for f of the form $\mu(r) = r + O(1)$, giving

$$
\begin{aligned}
\dim(x \mid y) &= \liminf_{r\to\infty} \frac{K_{r|r}(x \mid y)}{r} \\
&= \liminf_{r\to\infty} \frac{K_{r|r+O(1)}(x \mid y)}{r} && \text{[Lemma 1.7.3(ii)]} \\
&= \liminf_{r\to\infty} \frac{K_{r|\mu(r+1)}(x \mid y)}{r} \\
&\leq \liminf_{r\to\infty} \frac{K_{r|r}(x \mid f(y))}{r} \\
&= \dim(x \mid f(y)).
\end{aligned}
$$

□

Analogous results hold for when the inverse of a function is computable and uniformly continuous.

Lemma 2.2.3 (Reverse Modulus and Data Processing Inequalities for Conditional Complexity)**.** *Let $m, n, k \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \omega$.*

(i) If $f : \mathbb{R}^m \to \mathbb{R}^k$ is a co-computable, co-uniformly continuous map with computable inverse-modulus μ, then

$$K_{r|s}(x \mid y) \leq K_{\mu(r+1)|s}(f(x) \mid y).$$

In particular, if f is also co-Lipschitz continuous,

$$\dim(x \mid y) \leq \dim(f(x) \mid y).$$

(ii) If $f : \mathbb{R}^n \to \mathbb{R}^k$ is is a co-computable, co-uniformly continuous map with computable

inverse-modulus μ, then

$$K_{r|\mu(s+1)}\left(x \mid f(y)\right) \leq K_{r|s}\left(x \mid y\right).$$

In particular, if f is also co-Lipschitz continuous,

$$\dim(x \mid f(y)) \leq \dim(x \mid y).$$

Proof. Both parts follow by applying the corresponding result in the previous lemma.

(i) Denote $z = f(x) \in \mathbb{R}^k$. Apply Lemma 2.2.2(i) to f^{-1} and $(f^{-1}(z), y)$.

(ii) Denote $z = f(y) \in \mathbb{R}^k$. Apply Lemma 2.2.2(ii) to f^{-1} and $(x, f^{-1}(z))$.

□

Theorem 2.2.4. *Let $m, n \in \omega$, $x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$.*

(i) If $f : \mathbb{R}^m \to \mathbb{R}^m$ is a bi-computable, bi-Lipschitz continuous map, then

$$K_{r|s}(f(x) \mid y) = K_{r|s}(x \mid y); \quad \textit{so,} \quad \dim(f(x) \mid y) = \dim(x \mid y).$$

(ii) If $f : \mathbb{R}^n \to \mathbb{R}^n$ a bi-computable, bi-Lipschitz continuous map, then

$$K_{r|s}(x \mid f(y)) = K_{r|s}(x \mid y); \quad \textit{so,} \quad \dim(x \mid f(y)) = \dim(x \mid y).$$

Proof. We combine Lemmas 2.2.2 and 2.2.3,

(i) That f is both Lipschitz continuous and co-Lipschitz continuous implies:

$$\begin{aligned} K_{r|s}(f(x) \mid y) &\leq K_{r+O(1)|s}(x \mid y) && \text{[Lemma 2.2.2(i)]} \\ &\leq K_{r|s}(x \mid y) && \text{[Lemma 1.7.3(i)]} \\ &\leq K_{r+O(1)|s}(f(x) \mid y) && \text{[Lemma 2.2.3(i)]} \\ &\leq K_{r|s}(f(x) \mid y) && \text{[Lemma 1.7.3(i)].} \end{aligned}$$

(ii) That f is both Lipschitz continuous and co-Lipschitz continuous implies:

$$\begin{aligned} K_{r|s}(x \mid f(y)) &\leq K_{r|s+O(1)}(x \mid f(y)) && \text{[Lemma 1.7.3(ii)]} \\ &\leq K_{r|s}(x \mid y) && \text{[Lemma 2.2.3(ii)]} \end{aligned}$$

$$
\begin{aligned}
&\leq K_{r|s+O(1)}(x \mid y) && \text{[Lemma 1.7.3(ii)]} \\
&\leq K_{r|s}(x \mid f(y)) && \text{[Lemma 2.2.2(ii)].}
\end{aligned}
$$

□

By the properties of Hausdorff dimension, fractal geometry is considered to be the study of fractal properties invariant under the group of bi-Lipschitz continuous transformations [23]. Given the previous result, we may say something similar for effectivized fractal geometry: we study fractal properties which are invariant under the group of bi-computable, bi-Lipschitz continuous transformations [52].

Chapter 3 | Continuous, Absolutely Lipschitz Families

3.1 Background

It is an interesting and difficult problem to describe how effective Hausdorff dimension behaves as a function over Euclidean space. One approach towards a partial characterization involves tracking the effective dimension along function graphs. For instance, given a function $f : \mathbb{R} \to \mathbb{R}$, we ask how $\dim(x, f(x))$ might relate to $\dim(x)$ and f? If f is both computable and Lipschitz continuous, Lemma 2.2.2(i) implies $\dim(x, f(x)) = \dim(x)$. But when f is not computable nor Lipschitz continuous, showing a relation is more difficult. It may be that f is parameterized by some real tuple $\boldsymbol{\alpha}$ and computable as a map given $\boldsymbol{\alpha}$ as an oracle. What sort of uniform continuity properties should f possess in order to relate $\dim(x, f(x))$ back to the dimensions of x and the parameters in $\boldsymbol{\alpha}$?

The following theorem was originally proved in 2017 by N. Lutz and D. Stull [40]. It states that the effective dimension of a point on a line in the Euclidean plane may be bounded from below by a sum of terms depending on the effective dimensions of the input, slope, and intercept.

Theorem 3.1.1 (Theorem 1 in [40])**.** *For every $a, b, x \in \mathbb{R}$ and $B \in 2^{\leq\omega}$,*

$$\dim^B(x, ax + b) \geq \dim^B(x|a, b) + \min\left\{\dim^B(a, b), \dim^{B,a,b}(x)\right\}. \tag{3.1}$$

In particular, for almost every $x \in \mathbb{R}$, $\dim(x, ax + b) = 1 + \min\{\dim(a, b), 1\}$.

This is the prototypical example for relating $\dim(x, f(x))$ to the dimensions of x and the parameterization of f. Lutz and Stull employed prefix Kolmogorov complexity to prove this result. And as a consequence, they achieved a better lower bound on the classical Hausdorff dimension of generalized sets of Furstenberg type than were known at the time (see Theorem 4.3 of [40]). The bound follows from an application of the Point-to-Set Principle 1.12.1.

The original proof method for Theorem 3.1.1 might be described as following a sequence of three lemmas:

- **Inverse Lemma**: Lower-bounding the complexity of a point found in the intersection of many sufficiently complex tubes.
- **Enumeration Lemma**: Lower-bounding the complexity of tubes passing through a fixed point.
- **Oracle Construction**: Building an oracle with knowledge about a fixed tube passing through a fixed point, and not much more.

A simplified proof of Theorem 3.1.1 might first attempt to "frontload" the Kolmogorov complexity methods to achieve a *finitary* theorem on finite binary strings. Then, one could pass to the limit-inferior and obtain an *infinitary* theorem about effective dimensions. This sort of separation of effective methods from the approximation step is made possible by the results covered in Section 2.1. Initially, A. Case and J. Lutz proved prefix complexity could be approximated by K-minimizers: see Lemma 4.9 of [6]. We confirmed this for conditional complexity as well in Lemma 2.1.9. Together, N. Lutz and D. Stull showed that conditional complexity is also approximated by dyadic truncations: see Lemma A.1 (1.7.4) and Lemma A.3 (1.7.5) of [40].

A finitary result is not totally straightforward. The following example by Alexander Shen makes this apparent. Let us call two finitary objects u and v to be *independently random* if:

$$K(u, v) = K(u) + K(v) = \text{len}(u) + \text{len}(v),$$

where each equality holds up to logarithmic terms in the lengths of u and v.

Example 3.1.2. Fix $r \in \omega$ and the r-dyadic reals $a, b, x \in \mathbb{D}_r$ as follows. Take,

$$a = 0.\underbrace{0 \cdots 0}_{\frac{r}{2}\text{-many}}\bar{a}_{\text{rand}}, \quad \text{and} \quad b = 0.\underbrace{0 \cdots 0}_{\frac{r}{2}\text{-many}}\bar{b}_{\text{rand}},$$

where the binary strings $\bar{a}_{\text{rand}}, \bar{b}_{\text{rand}} \in 2^{\lfloor r/2 \rfloor}$ are independently random. And select x to be independently random with respect to the pair $\langle a, b \rangle$.

It remains to see whether a, b, and x satisfy a finitary version of (3.1). Denote by $y = (ax + b) \upharpoonright r$. We ask whether

$$K(x, y) \overset{?}{\geq} K(x \mid a, b) + \min \{K(a, b), K(x \mid a, b)\}. \tag{3.2}$$

Note that $K(y) \leq \frac{r}{2}$ holds up to logarithmic terms in r, for the first approximately $\frac{r}{2}$-bits in the binary expansion of y will be zero. Again dropping any logarithmic terms in r, we may calculate:

$$\begin{aligned} K(x, y) \leq K(x) + K(y) \leq r + \frac{r}{2} = \frac{3r}{2} \\ < 2r = r + r = K(x \mid a, b) + \min \{K(a, b), K(x \mid a, b)\}. \end{aligned}$$

So, (3.2) fails for such a choice of a, b, and x for large enough r.

This example illustrates the main issue in obtaining a finitary theorem from which to derive Theorem 3.1.1: the line's parameters (or input) may not consistently achieve high complexity across their truncations, possibly producing too simple of an output. Theorem 3.3.3 below only applies to finitary parameters and inputs whose truncations grow at least as fast as some consistent rate.

We further consider the continuity and algorithmic properties of the family of planar lines which make Theorem 3.1.1 possible. Any planar line f may be represented in slope-intercept form: $f(x) = ax + b$, meaning the finite tuple $(a, b) \in \mathbb{R}^2$ serves as a parameterization of f under slope-intercept form. Let (u, v) be the slope-intercept representation of another planar line g. The first geometric property is referred to as the *Intersecting Tubes Lemma*, which approximately states that whenever $f(x) = g(x)$ yet $a \neq u$, one may approximate x by the quantity $\frac{b-v}{u-a}$ up to a precision that improves as a and u become more orthogonal as slopes. This could be seen as a consequence of linear functions being co-Lipschitz continuous. Second, when represented in slope-intercept form, the function f is nothing more than the composition of a single multiplication followed by a single addition on some input: $f(x) = ax + b$. Lines thereby have the following property: one may use rational approximations to a, b, and x to compute a similarly precise approximation to the output, $f(x)$. This property is a bit weaker than Lipschitz continuity for f, since the precision to which we can approximate $f(x)$ will also depend (linearly) on the sizes of $|a|$ and $|x|$. We investigate whether any function family which is sufficiently Lipschitz and co-Lipschitz continuous as above should also admit a lower bound result of the same form as in Theorem 3.1.1.

3.2 Computable Absolutely Lipschitz Families

3.2.1 Definition of a CALF

We are interested in capturing the computability and algebraic properties of families of curves or surfaces like the planar lines. So we will only consider maps of the form $\Phi : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n$, where the first m components of the arguments encode a parameterization of some function in the family, and the following ℓ components serve as input. Given any such Φ and open subsets $\Omega \subseteq \mathbb{R}^m$ and $\Xi \subseteq \mathbb{R}^\ell$ (which we call our *domains*), we might define the family of all parameterized, real-valued functions under Φ on domain $\Omega \times \Xi$:

$$\mathcal{F}_\Phi(\Omega \times \Xi) = \{\Phi^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Omega\},$$

where $\Phi^{\boldsymbol{\alpha}} : \Xi \to \mathbb{R}^n$ denotes the map $x \mapsto \Phi(\boldsymbol{\alpha}, x)$. And since we are also interested in statements about finitary inputs, we further denote for any $r \in \omega$,

$$\Omega_r := \Omega \upharpoonright r = \{\boldsymbol{\alpha} \upharpoonright r : \boldsymbol{\alpha} \in \Omega\} \subseteq \mathbb{D}_r^m,$$

and let $\Omega_{<\omega}$ be the union of all $\Omega_r \subseteq \mathbb{D}^m$. Analogous notation will apply for Ξ.

Since the domain Ω is assumed to be open, we have for any $\boldsymbol{\alpha} \in \Omega$ that there exists $r \gg 1$ for which $\boldsymbol{\alpha} \upharpoonright r \in \Omega_r \cap \Omega$. An analogous statement holds for Ξ. This will be useful for reducing our infinitary theorem to the finitary case.

Locally, certain parameter components may or may not (linearly) depend on the input x. Let us fix a way to distinguish between these two behaviors. Fix any two parameters $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega \subseteq \mathbb{R}^m$. A component $0 \leq i < m$ will be considered *roughly constant* for Φ at $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ if, for each $\boldsymbol{\alpha}', \boldsymbol{\beta}'$ which differ from $\boldsymbol{\alpha}, \boldsymbol{\beta}$ only at component i, and for all $x \in \Xi$,

$$\left|\left|(\Phi^{\boldsymbol{\alpha}'}(x) - \Phi^{\boldsymbol{\beta}'}(x)) - (\Phi^{\boldsymbol{\alpha}}(x) - \Phi^{\boldsymbol{\beta}}(x))\right|\right| \leq O\left(||\boldsymbol{\alpha}' - \boldsymbol{\alpha}|| + ||\boldsymbol{\beta}' - \boldsymbol{\beta}||\right)$$

independently of x. Otherwise, the component i will be considered *co-Lipschitz* for Φ. We collect into the tuple $(\boldsymbol{\alpha} - \boldsymbol{\beta})_{\mathrm{cL}}$ all differences $\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i$ where i is a co-Lipschitz component for Φ at $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Definition 3.2.1. A *computable absolutely Lipschitz family (CALF)* on a domain $\Omega \times \Xi \subseteq \mathbb{R}^m \times \mathbb{R}^\ell$ is a partial-computable map $\Phi : \Omega \times \Xi \to \mathbb{R}^n$ such that

(i) Φ is *scaling Lipschitz continuous*. That is, for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega$ and $x_1, x_2 \in \Xi$,

$$\left|\left|\Phi^{\boldsymbol{\alpha}}(x_1) - \Phi^{\boldsymbol{\beta}}(x_2)\right|\right| \leq (O(||x_1||) + O(1) + o(1)) \cdot ||\boldsymbol{\alpha} - \boldsymbol{\beta}|| \\ + (O(||\boldsymbol{\alpha}||) + O(1)) \cdot ||x_1 - x_2|| \,,$$

where the $o(1)$ term vanishes as $||x_1 - x_2|| \to 0$.

(ii) Φ has *scaling co-Lipschitz continuous differences*. That is, for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega_{<\omega}$, it holds that $\Phi^{\boldsymbol{\alpha}} - \Phi^{\boldsymbol{\beta}}$ is either constant or *scaling co-Lipschitz continuous*, meaning for all $x_1, x_2 \in \Xi$,

$$||(\boldsymbol{\alpha} - \boldsymbol{\beta})_{\text{cL}}|| \cdot ||x_1 - x_2|| = O_{||x_1||,||x_2||}\left(\left|\left|(\Phi^{\boldsymbol{\alpha}}(x_1) - \Phi^{\boldsymbol{\beta}}(x_1)) - (\Phi^{\boldsymbol{\alpha}}(x_2) - \Phi^{\boldsymbol{\beta}}(x_2))\right|\right|\right),$$

where the constant may depend on $||x_1||$ and $||x_2||$; and there is an algorithm deciding this from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

(iii) Φ has *dense intersections*. That is, for all $r \in \omega$, $\boldsymbol{\alpha} \in \Omega_r$, and $x \in \Xi$, there is an algorithm using $x \upharpoonright r$ and $\Phi^{\boldsymbol{\alpha}}(x) \upharpoonright r$ (as well as r, m, n, ℓ, and $\boldsymbol{\alpha} \upharpoonright O(1)$) to compute a $(2^{-r} \cdot O_{||\boldsymbol{\alpha}||,||x||,m,n}(1))$-approximation to some $\boldsymbol{\beta} \in B_{2^{-r} \cdot O_{||\boldsymbol{\alpha}||,||x||,m,n}(1)}(\boldsymbol{\alpha}) \cap \Omega_r$ satisfying $\Phi^{\boldsymbol{\alpha}} - \Phi^{\boldsymbol{\beta}}$ is scaling co-Lipschitz continuous with $-\log_2 ||(\boldsymbol{\alpha} - \boldsymbol{\beta})_{\text{cL}}|| \leq r - O_{||\boldsymbol{\alpha}||,||x||,m,n}(1)$.

Definition 3.2.2. Let Φ be a CALF on $\Omega \times \Xi$. We fix a measure of similarity between two parameters $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega$ as follows:

$$S(\boldsymbol{\alpha}, \boldsymbol{\beta}) = -\log ||(\boldsymbol{\alpha} - \boldsymbol{\beta})_{\text{cL}}|| \,.$$

In particular, if $\boldsymbol{\alpha} \upharpoonright r = \boldsymbol{\beta} \upharpoonright r$, then $S(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq r - O(1)$.

3.2.2 Standard Example of a CALF

Example 3.2.3 (Points on Planar Lines)**.** Take Φ to be the computable map on $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by

$$\Phi(a, b, x) = ax + b.$$

We show that Φ is a CALF on its domain. The real-function family $\mathcal{F}_\Phi$ associated to this map consists of all the non-vertical, planar lines as expressed in slope-intercept form with parameters a and b and input x. Let us fix two sets of slope-intercept parameters $(a, b), (u, v) \in \mathbb{R}^2$ and two inputs $x_1, x_2 \in \mathbb{R}$.

It is easy to see that Φ is scaling Lipschitz continuous: we may check using the triangle inequality that:

$$\left|\Phi^{(a,b)}(x_1) - \Phi^{(u,v)}(x_2)\right| \leq ||(a,b) - (u,v)|| \cdot (|x_1| + 1 + |x_1 - x_2|) + |a| \cdot |x_1 - x_2| .$$

Moreover, $\Phi^{(a,b)} - \Phi^{(u,v)}$ is constant if and only if $a = u$. In addition, the slope parameter is a co-Lipschitz parameter, while the intercept parameter is roughly constant in the sense defined above. Indeed, given different choices $a', b', u', v' \in \mathbb{R}$ and any $x \in \mathbb{R}$,

$$\left|(\Phi^{(a',b)}(x) - \Phi^{(u',v)}(x)) - (\Phi^{(a,b)}(x) - \Phi^{(u,v)}(x))\right| = |(a' - a) - (u' - u)| \, |x| ,$$
$$\left|(\Phi^{(a,b')}(x) - \Phi^{(u,v')}(x)) - (\Phi^{(a,b)}(x) - \Phi^{(u,v)}(x))\right| \leq |(b' - b) - (v' - v)| \leq |b' - b| + |v' - v| .$$

Thus, in this example, $||((a,b) - (u,v))_{\mathrm{cL}}|| = |a - u|$. It is also straightforward to check that:

$$\left|(\Phi^{(a,b)}(x_1) - \Phi^{(u,v)}(x_1)) - (\Phi^{(a,b)}(x_2) - \Phi^{(u,v)}(x_2))\right| = |a - u| \cdot |x_1 - x_2| .$$

So we also conclude that Φ has scaling co-Lipschitz continuous differences.

We finally argue that Φ has dense intersections. Fix $r \in \omega$, $(a, b) \in \mathbb{D}_r^2$, and $x \in \mathbb{R}$. We appeal to the the machine construction and geometric observations used in proving Lemma 6(i) of [40]. Their argument can be seen as constructing a Turing machine which receives a description for $(x \upharpoonright r, (ax + b) \upharpoonright r)$, along with descriptions for r and $(a, b) \upharpoonright 1$, to produce parameters $(u_0, v_0) \in B_{2^{-r}}(a, b)$ satisfying:

$$|u_0 \cdot (x \upharpoonright r) + v_0 - (ax + b) \upharpoonright r| < 2^{-r} \cdot (|u_0| + |x \upharpoonright r| + 3).$$

By their Observation A.4(ii), there exists $(u, v) \in B_{2^{\gamma - r}}(u_0, v_0) \cap \mathbb{D}_r^2$ such that $u \neq a$, where $\gamma = \log(2\,|a| + |x| + 5)$. These parameters (u, v) will necessarily agree in output with (a, b) at input x up to precision $2^{-r} \cdot O(|a| + |x| + 1)$.

3.3 Finitary Theorem

The following geometric observation is known as the *Intersecting Tubes Lemma* in the case of planar lines: that the closer to perpendicular two intersecting lines are to one another, the more precisely one may describe their point of intersection. For us, that generalizes to a statement of the form: the more similar the co-Lipschitz parameters of two Φ-curves are, the more precisely one may describe an input on which they approximately intersect.

Observation 3.3.1 (C.f. Observation A.5 in [40])**.** Take $\Phi : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n$ to be a CALF on domain $\Omega \times \Xi$. Fix $r \in \omega$, $x \in \Xi_r$, and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega_r$ with $\Phi^{\boldsymbol{\alpha}} - \Phi^{\boldsymbol{\beta}}$ being scaling co-Lipschitz continuous. If $\left|\left|\Phi^{\boldsymbol{\alpha}}(x) - \Phi^{\boldsymbol{\beta}}(x)\right|\right| \leq 2^{-r+O_n(1)}$, $k = S(\boldsymbol{\alpha}, \boldsymbol{\beta})$, and $r \geq k + O_{||x||,n}(1)$, then there is an algorithm approximating x to precision $2^{-r+k+O_{x,n}(1)}$ uniformly in $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and r.

Proof. Whenever another $x' \in \Xi$ satisfies $\left|\left|\Phi^{\boldsymbol{\alpha}}(x') - \Phi^{\boldsymbol{\beta}}(x')\right|\right| < 2^{-r+O_n(1)}$, then since Φ has co-Lipschitz continuous differences,

$$\begin{aligned} ||(\boldsymbol{\alpha} - \boldsymbol{\beta})_{\mathrm{cL}}|| \cdot ||x - x'|| &\leq O_{||x||,||x'||}(1) \cdot \left|\left|(\Phi^{\boldsymbol{\alpha}} - \Phi^{\boldsymbol{\beta}})(x) - (\Phi^{\boldsymbol{\alpha}} - \Phi^{\boldsymbol{\beta}})(x')\right|\right| \\ &\leq 2 \cdot O_{||x||,||x'||}(1) \cdot 2^{-r+O_n(1)}. \end{aligned} \tag{3.3}$$

So, $||x - x'|| \leq 2^{-r+k+O_{||x||,||x'||,n}(1)}$. This informs our search: splitting $\mathbb{R}^\ell$ into cubes with side-length 2^{-r}, we iterate through all dyadic rational $q \in \mathbb{D}_r^\ell$ and ask whether $\left|\left|\Phi^{\boldsymbol{\alpha}}(q) - \Phi^{\boldsymbol{\beta}}(q)\right|\right| < 2^{-r+O_{||q||,n}(1)}$. In light of (3.3), the search must terminate on a test point $q \in B_{2^{-r+k+O_{||x||,n}(1)}}(x)$. □

As a corollary to Observation 3.3.1, we have the following fact for any two curves with parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ which approximately agree on some input x. Essentially, $\boldsymbol{\beta}$ can simultaneously describe the portion of the co-Lipschitz components of $\boldsymbol{\alpha}$ shared with $\boldsymbol{\beta}$ along with a complementary portion of the bits of x.

Corollary 3.3.2 (C.f. Lemma 7 in [40])**.** *Take $\Phi : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n$ to be a CALF on domain $\Omega \times \Xi$. Fix $r \in \omega$, $x \in \Xi_r$, and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega_r$ with $\Phi^{\boldsymbol{\alpha}} - \Phi^{\boldsymbol{\beta}}$ being scaling co-Lipschitz continuous. If $\left|\left|\Phi^{\boldsymbol{\alpha}}(x) - \Phi^{\boldsymbol{\beta}}(x)\right|\right| \leq 2^{-r+O_n(1)}$ and $k = S(\boldsymbol{\alpha}, \boldsymbol{\beta})$, then for each $r \geq k + O_{||x||,n}(1)$,*

$$K(\boldsymbol{\beta}) \geq K(\boldsymbol{\alpha} \upharpoonright k) + K(x \upharpoonright (r - k) \mid \boldsymbol{\alpha}) - O_{||\boldsymbol{\alpha}||,||x||,n}(\log r).$$

Proof. Suppose $r \geq k + O_{||x||,n}(1)$ from Observation 3.3.1. We perform symmetry of information 1.6.1 several times to confirm:

$$\begin{aligned} K(\boldsymbol{\beta}) &\geq K(\boldsymbol{\beta} \mid \boldsymbol{\alpha}) + [K(\boldsymbol{\alpha}) - K(\boldsymbol{\alpha} \mid \boldsymbol{\beta})] - O(\log r) \\ &\geq K(x \upharpoonright (r-k) \mid \boldsymbol{\alpha}) + [K(\boldsymbol{\alpha}) - K(\boldsymbol{\alpha} \mid \boldsymbol{\beta} \upharpoonright k)] - O(\log r) \quad \text{[Observation 3.3.1]} \\ &\geq K(x \upharpoonright (r-k) \mid \boldsymbol{\alpha}) + [K(\boldsymbol{\alpha}) - K(\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \upharpoonright k)] - O(\log r) \\ &= K(x \upharpoonright (r-k) \mid \boldsymbol{\alpha}) + K(\boldsymbol{\alpha} \upharpoonright k) - O(\log r). \end{aligned}$$

□

Theorem 3.3.3 (Finitary Lower Bound)**.** *Take $\Phi : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n$ to be a CALF on domain $\Omega \times \Xi$. Let $d < \delta \in [0, \ell]$, $r \in \omega$. Suppose $\boldsymbol{\alpha} \in \Omega_r$ and $x \in \Xi_r$ satisfy for each $k \leq r$,*

(i) $K(\boldsymbol{\alpha} \upharpoonright k) \geq dk - o(k)$,

(ii) $K(x \upharpoonright k \mid \boldsymbol{\alpha}) \geq \delta k - o(k)$.

Then,

$$K(x, \Phi^{\boldsymbol{\alpha}}(x)) \geq K(\boldsymbol{\alpha}, x) - m \cdot \frac{K(\boldsymbol{\alpha}) - dr}{\delta - d} - o(r).$$

Proof. Let m, n, ℓ, d, δ, r, $\boldsymbol{\alpha}$, and x be as in the statement. Further denote by $y := \Phi^{\boldsymbol{\alpha}}(x) \upharpoonright r \in \mathbb{D}^n_r$ the level-r approximation to the output of $\Phi^{\boldsymbol{\alpha}}$ at x. Since Φ has density of intersections, we may fix $\boldsymbol{\beta} \in \Omega_r$ such that $\Phi^{\boldsymbol{\alpha}} - \Phi^{\boldsymbol{\beta}}$ is scaling co-Lipschitz continuous, $||\boldsymbol{\alpha} - \boldsymbol{\beta}|| \leq 2^{-r+O_{||\boldsymbol{\alpha}||,||x||,m,n}(1)}$, and $S(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq r - O_{||\boldsymbol{\alpha}||,||x||,m,n}(1)$. And since Φ is scaling Lipschitz continuous, this implies that the outputs $\Phi^{\boldsymbol{\alpha}}(x)$ and $\Phi^{\boldsymbol{\beta}}(x)$ agree to a similar precision:

$$\left|\left|\Phi^{\boldsymbol{\alpha}}(x) - \Phi^{\boldsymbol{\beta}}(x)\right|\right| \leq O(||\boldsymbol{\alpha}|| + |x| + 1) \cdot ||\boldsymbol{\alpha} - \boldsymbol{\beta}|| \leq 2^{-r+\gamma},$$

where $\gamma = 2^{O_{||\boldsymbol{\alpha}||,||x||,m,n}(1)}$.

Furthermore, by the density of intersections assumption, we see:

$$K((\boldsymbol{\beta}, x) \upharpoonright (r - \gamma)) \leq K(x, y) + K_1(\boldsymbol{\alpha}) + K(r) + K(m, n, \ell) + O(1). \tag{3.4}$$

Let $k = S(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and recall $r \geq k + O_{||\boldsymbol{\alpha}||,||x||,m,n}(1)$. Since $\boldsymbol{\alpha}$ can approximately compute $\boldsymbol{\beta}$, we deduce the following numerical condition:

$$\begin{aligned}
K(\boldsymbol{\alpha}) &\geq K(\boldsymbol{\beta} \upharpoonright (r-\gamma)) - o(r) && [\boldsymbol{\beta} \in B_{2^{\gamma-r}}(\boldsymbol{\alpha})] \\
&\geq K(\boldsymbol{\beta}) - o(r) && [\gamma = O_{||\boldsymbol{\alpha}||,||x||,m,n}(1)] \\
&\geq K(\boldsymbol{\alpha} \upharpoonright k) + K(x \upharpoonright (r-k) \mid \boldsymbol{\alpha}) - o(r) && [\text{Corollary 3.3.2}] \\
&\geq dk + \delta(r-k) - o(r) && [\text{Assumptions (i) \& (ii)}] \\
&= dr + (\delta - d)(r-k) - o(r),
\end{aligned}$$

We rearrange to see:

$$r - k \leq \frac{K(\boldsymbol{\alpha}) - dr}{\delta - d} + o(r), \tag{3.5}$$

which is well-defined as $\delta > d$. Notice by assumption that $B_{2^{-r}}(\boldsymbol{\beta}) \subseteq B_{2^{-k+o(r)}}(\boldsymbol{\alpha})$. We use this fact and the Chain Rule 1.6.1 to conclude:

$$\begin{aligned}
K(x,y) &\geq K(\boldsymbol{\beta}, x) - o(r) && [(3.4)] \\
&\geq K(\boldsymbol{\alpha} \upharpoonright k, x) - o(r) \\
&= K(\boldsymbol{\alpha} \upharpoonright k \mid x) + K(x) - o(r) \\
&\geq K(\boldsymbol{\alpha} \mid x) - m \cdot (r-k) + K(x) - o(r) \\
&\geq K(\boldsymbol{\alpha}, x) - m \cdot \frac{K(\boldsymbol{\alpha}) - dr}{\delta - d} - o(r). && [(3.5)]
\end{aligned}$$

□

3.4 Infinitary Theorem

We leverage the finitary theorem 3.3.3 to prove the corresponding result about the effective dimension of points on the graph of a map in the family given by a CALF. The last component we require to do this is the oracle construction developed by N. Lutz and D. Stull in [40].

Lemma 3.4.1 (Lemma 8 of [40])**.** *Fix $m, r \in \omega$, $\mathbf{x} \in \mathbb{R}^m$, and rational $0 \leq \eta \leq \dim(\mathbf{x})$. Then there exists an oracle $X = X(\mathbf{x}, m, r, \eta)$ satisfying:*

(i) For each $k \leq r$,

$$K_k^X(\mathbf{x}) = \min\{\eta r, K_k(\mathbf{x})\} \pm O(\log r).$$

(ii) For any $n, s \in \omega$ and $\mathbf{y} \in \mathbb{R}^n$,

$$K^X_{s|r}(\mathbf{y} \mid \mathbf{x}) = K_{s|r}(\mathbf{y} \mid \mathbf{x}) \pm O(\log r),$$
$$K^{\mathbf{x},X}_s(\mathbf{y}) = K^{\mathbf{x}}_s(\mathbf{y}) \pm O(\log r).$$

Part (i) of the construction claims X can simplify the first r many dyadic truncations of $\mathbf{x}$ to not exceed the limiting information density of $\mathbf{x}$, whereas part (ii) ensures that X cannot meaningfully aid $\mathbf{x}$ in making computations.

Theorem 3.4.2 (Infinitary Lower Bound)**.** *Let $\Phi : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n$ be a CALF on domain $\Omega \times \Xi$. Then for every $\boldsymbol{\alpha} \in \Omega$, $x \in \Xi$, and $B \in 2^{\leq\omega}$,*

$$\dim^B(x, \Phi^{\boldsymbol{\alpha}}(x)) \geq \dim^B(x|\boldsymbol{\alpha}) + \min\left\{\dim^B(\boldsymbol{\alpha}), \dim^{B,\boldsymbol{\alpha}}(x)\right\}.$$

Proof. The following argument may be relativized to an arbitrary oracle $B \in 2^{\leq\omega}$. Fix $\boldsymbol{\alpha} \in \Omega$ and $x \in \Xi$. The claim of the theorem is trivial when $\dim(x \mid \boldsymbol{\alpha}) = 0$ (in which case, by Lemma 1.7.7, $\dim^{\boldsymbol{\alpha}}(x) = 0$ as well). So consider the case of $\delta := \dim^{\boldsymbol{\alpha}}(x) > 0$, and let

$$d \in \mathbb{Q} \cap [0, \dim(\boldsymbol{\alpha})] \cap [0, \dim^{\boldsymbol{\alpha}}(x)).$$

Clearly, $\delta > d$. Fix $r \gg 1$ sufficiently large so that $\boldsymbol{\alpha} \upharpoonright r \in \Omega$ and $x \upharpoonright r \in \Xi$. Let $X_r := X(\boldsymbol{\alpha}, m, r, d)$ be an oracle for $\boldsymbol{\alpha}$ at information density d as guaranteed by Lemma 3.4.1. We now check that we may apply Theorem 3.3.3 to $\boldsymbol{\alpha} \upharpoonright r$ and $x \upharpoonright r$ at precision-level r. Let $0 \leq k \leq r$. We skip writing $(\boldsymbol{\alpha} \upharpoonright r) \upharpoonright k = \boldsymbol{\alpha} \upharpoonright k$ and similar for x.

Then, the assumption (i) of Theorem 3.3.3 follows from Lemma 1.7.4:

$$K(\boldsymbol{\alpha} \upharpoonright k) = K_k(\boldsymbol{\alpha}) \pm o(k) \geq \dim(\boldsymbol{\alpha})k - o(k) \geq dk - o(k).$$

And the assumption (ii) similarly follows from Lemma 1.7.4 relativized to $\boldsymbol{\alpha}$:

$$K^{\boldsymbol{\alpha}}(x \upharpoonright k) = K^{\boldsymbol{\alpha}}_k(x) \pm o(k) \geq \dim^{\boldsymbol{\alpha}}(x)k - o(k) = \delta k - o(k).$$

Therefore, we may apply Theorem 3.3.3 to $\boldsymbol{\alpha} \upharpoonright r$ and $x \upharpoonright r$. Note that if $\boldsymbol{\alpha}_r = \boldsymbol{\alpha} \upharpoonright r$ and $x_r = x \upharpoonright r$, then $K(\Phi^{\boldsymbol{\alpha}}(x) \upharpoonright r) = K(\Phi^{\boldsymbol{\alpha}_r}(x_r) \upharpoonright r) \pm o(r)$. This follows from Φ being scaling Lipschitz continuous. So,

$$\begin{aligned}
\dim(x, \Phi^{\boldsymbol{\alpha}}(x)) &\geq \dim^{X_r}(x, \Phi^{\boldsymbol{\alpha}}(x)) \\
&= \liminf_{r\to\infty} \frac{1}{r} K_r^{X_r}(x, \Phi^{\boldsymbol{\alpha}}(x)) \\
&= \liminf_{r\to\infty} \frac{1}{r} K^{X_r}((x, \Phi^{\boldsymbol{\alpha}}(x)) \upharpoonright r) && \text{[Lemma 1.7.4 rel. to } X_r\text{]} \\
&= \liminf_{r\to\infty} \frac{1}{r} K^{X_r}(x_r, \Phi^{\boldsymbol{\alpha}_r}(x_r) \upharpoonright r) && \text{[}\Phi\text{ is scaling Lipschitz]} \\
&\geq \liminf_{r\to\infty} \frac{1}{r} \left[K^{X_r}(\boldsymbol{\alpha}_r, x_r) - m \cdot \frac{K^{X_r}(\boldsymbol{\alpha}_r) - dr}{\delta - d} \right] && \text{[Theorem 3.3.3 rel to } X_r\text{]} \\
&= \liminf_{r\to\infty} \frac{1}{r} \left[K_r^{X_r}(\boldsymbol{\alpha}, x) - m \cdot \frac{K_r^{X_r}(\boldsymbol{\alpha}) - dr}{\delta - d} \right] && \text{[Lemma 1.7.4 rel to } X_r\text{]} \\
&\geq \liminf_{r\to\infty} \frac{1}{r} K_r^{X_r}(\boldsymbol{\alpha}, x) && \text{[Lemma 3.4.1(i)]} \\
&\geq \liminf_{r\to\infty} \frac{1}{r} \left[K_{r|r}^{X_r}(x \mid \boldsymbol{\alpha}) + K_r^{X_r}(\boldsymbol{\alpha}) \right] && \text{[Theorem 1.7.6]} \\
&= \liminf_{r\to\infty} \frac{1}{r} \left[K_{r|r}(x \mid \boldsymbol{\alpha}) + dr \right] && \text{[Lemma 3.4.1(i),(ii)]} \\
&= \dim(x|\boldsymbol{\alpha}) + d.
\end{aligned}$$

As d was an arbitrary rational number in $[0, \dim(\boldsymbol{\alpha})] \cap [0, \dim^{\boldsymbol{\alpha}}(x))$, we conclude:

$$\dim(x, \Phi^{\boldsymbol{\alpha}}(x)) \geq \dim(x \mid \boldsymbol{\alpha}) + \min\{\dim(\boldsymbol{\alpha}), \dim^{\boldsymbol{\alpha}}(x)\}.$$

□

Theorem 3.1.1 follows as a consequence of Theorem 3.4.2 as applied to the CALF Φ from Example 3.2.3.

3.5 Dimension Spectrum of CALF Maps

According to Theorem 3.1.1, if $\Phi^{\boldsymbol{\alpha}}$ is a map from a CALF, then points $(x, \Phi^{\boldsymbol{\alpha}}(x))$ on its graph are at least somewhat complex. Our goal in this section is to understand the effective dimension of these points as a distribution: e.g., can the lower bound in Theorem 3.1.1 be exact for most points along $\Phi^{\boldsymbol{\alpha}}$? The following results are direct generalizations of those from Section 4 of [65]. For the remainder of the section, we fix a CALF $\Phi : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n$ on a domain $\Omega \times \Xi$.

The first result is an observation that directly follows from the approximate chain rule 1.7.6 and the density of intersections of Φ.

Observation 3.5.1. For each $\boldsymbol{\alpha} \in \Omega$ and $x_1, x_2 \in \Xi$,

$$\begin{aligned} &K_r(\boldsymbol{\alpha}) + K_r(x_1 \mid \boldsymbol{\alpha}) + K_r(x_2 \mid \boldsymbol{\alpha}, x_1) - O(\log r) \\ &\quad \leq K_r(\boldsymbol{\alpha}, x_1, x_2) \\ &\quad \leq K_r(x_1, \Phi^{\boldsymbol{\alpha}}(x_1)) + K_r(x_2, \Phi^{\boldsymbol{\alpha}}(x_2)) + O_{||x_1||,||x_2||}(1) + 2\log ||x_1 - x_2|| . \end{aligned}$$

The final upper bound in Observation 3.5.1 involves two separate complexity terms which are not guaranteed to achieve their limiting lower densities at the same precision-levels, preventing us from immediately concluding a result relating the effective dimension of these objects. Yet, we may characterize the size of the set of instances when $K_r(x_1, \Phi^{\boldsymbol{\alpha}}(x_1))$ and $K_r(x_2, \Phi^{\boldsymbol{\alpha}}(x_2))$ are simultaneously minimal.

Lemma 3.5.2. *Let $\delta \in (0, \ell)$, $\boldsymbol{\alpha} \in \Omega$, $x_1 \in \Xi$, and $n, r \in \omega$ be such that $\frac{2}{\sqrt{r}} < \frac{1}{n}$. Assume that*

$$\delta \cdot r \leq K_r^{\boldsymbol{\alpha}}(x_1, \Phi^{\boldsymbol{\alpha}}(x_1)), \quad \textit{and} \quad K_r(x_1, \Phi^{\boldsymbol{\alpha}}(x_1)) \leq \delta \cdot r + \frac{K_r(\boldsymbol{\alpha})}{2} - \frac{r}{n}.$$

Then, for any other $x_2 \in \Xi$ satisfying $K_r(x_2, \Phi^{\boldsymbol{\alpha}}(x_2)) < \delta \cdot r + \frac{K_r(\boldsymbol{\alpha})}{2}$, it holds that either

$$-\log ||x_1 - x_2|| \leq \sqrt{r} + O_{||x_1||,||x_2||}(1) + O(\log r), \quad \textit{or} \quad K_r(x_2 \mid \boldsymbol{\alpha}, x_1) < \delta \cdot r.$$

In words, the following holds for all sufficiently large precision-levels. We suppose x_1 is sufficiently complex with respect to α, while the pair $(x_1, \Phi^{\boldsymbol{\alpha}}(x_1))$ is not much more complex. Then, for any other input x_2 with a simple $(x_2, \Phi^{\boldsymbol{\alpha}}(x_2))$, either x_2 is sufficiently far from x_1, or x_2 is simple to describe from $\boldsymbol{\alpha}$ and x_1.

Proof. Suppose the hypothesis holds but that neither condition is satisfied for some $x_2 \in \Xi$. Then by Observation 3.5.1,

$$\begin{aligned} &K_r(\boldsymbol{\alpha}) + 2\delta r - O(\log r) \\ &\quad \leq K_r(\boldsymbol{\alpha}) + K_r(x_1 \mid \boldsymbol{\alpha}) + K_r(x_2 \mid \boldsymbol{\alpha}, x_1) - O(\log r) \\ &\quad \leq K_r(x_1, \Phi^{\boldsymbol{\alpha}}(x_1)) + K_r(x_2, \Phi^{\boldsymbol{\alpha}}(x_2)) + O_{||x_1||,||x_2||}(1) + 2\log ||x_1 - x_2|| \\ &\quad < \left[\delta r + \frac{K_r(\boldsymbol{\alpha})}{2} - \frac{r}{n}\right] + \left[\delta r + \frac{K_r(\boldsymbol{\alpha})}{2}\right] + O_{||x_1||,||x_2||}(1) + 2\log ||x_1 - x_2|| . \end{aligned}$$

Since the first condition on x_2 fails, we deduce $\frac{1}{n} < \frac{2}{\sqrt{r}}$, a contradiction. $\square$

Given $\delta \in (0, \ell)$, denote by

$$D^{\boldsymbol{\alpha}}(\delta) := \left\{ x \in \Xi : \dim(x, \Phi^{\boldsymbol{\alpha}}(x)) < \delta + \frac{\dim(\boldsymbol{\alpha})}{2} \right\}$$

the set of inputs which map to the point $(x, \Phi^{\boldsymbol{\alpha}}(x))$ having effective dimension less than $\delta + \dim(\boldsymbol{\alpha})/2$. Now, we may state the main result for the dimension spectrum of $\Phi^{\boldsymbol{\alpha}}$.

Theorem 3.5.3. *For every* $\boldsymbol{\alpha} \in \Omega$ *and* $\delta \in (0, \ell)$*, we have* $\dim_{\mathrm{H}}(D^{\boldsymbol{\alpha}}(\delta)) \leq \delta$*.*

That is, very few points along the graph of $\Phi^{\boldsymbol{\alpha}}$ may be of small effective dimension, and almost every input x produces a point $(x, \Phi^{\boldsymbol{\alpha}}(x))$ of effective dimension at least ℓ.

Proof. Fix $\boldsymbol{\alpha} \in \Omega$ and $\delta \in (0, \ell)$. Really, we work with the sets:

$$D_n^{\boldsymbol{\alpha}}(\delta) := \left\{ x \in \Xi : (\exists^\infty r \in \omega) \left[K_r(x, \Phi^{\boldsymbol{\alpha}}(x)) < \delta \cdot r + \frac{K_r(\boldsymbol{\alpha})}{2} - \frac{r}{n} \right] \right\},$$

which cover $D^{\boldsymbol{\alpha}}(\delta)$, since if $x \in D^{\boldsymbol{\alpha}}(\delta)$ and $\varepsilon > 0$, there will exist infinitely many $r \in \omega$ for which

$$K_r(x, \Phi^{\boldsymbol{\alpha}}(x)) < \delta \cdot r + \frac{\dim(\boldsymbol{\alpha})}{2} \cdot r - \varepsilon \cdot r < \delta \cdot r + \frac{K_r(\boldsymbol{\alpha}) + o(r)}{2} - \varepsilon \cdot r,$$

meaning for sufficiently large values of $n \in \omega$, we have $x \in D_n^{\boldsymbol{\alpha}}(\delta)$. Therefore, by the countable stability of Hausdorff dimension, it suffices to show that $\dim_{\mathrm{H}}(D_n^{\boldsymbol{\alpha}}(\delta)) \leq \delta$ for all $n \in \omega$.

To begin, for each $r \in \omega$, fix a point $x_r \in \Xi$, if it exists, satisfying the hypotheses of Lemma 3.5.2, i.e.,

$$\delta \cdot r \leq K_r^{\boldsymbol{\alpha}}(x_r, \Phi^{\boldsymbol{\alpha}}(x_r)), \quad \text{and} \quad K_r(x_r, \Phi^{\boldsymbol{\alpha}}(x_r)) \leq \delta \cdot r + \frac{K_r(\boldsymbol{\alpha})}{2} - \frac{r}{n}.$$

And let $B \in 2^{\leq \omega}$ be an oracle encoding all the $(x_r)_{r \in \omega}$ which exist. We now consider three cases for a fixed $x \in D_n^{\boldsymbol{\alpha}}(\delta)$.

Suppose first that $K_r^{\boldsymbol{\alpha}}(x) \leq \delta \cdot r$ for infinitely many $r \in \omega$. Then, clearly, $\dim^{\boldsymbol{\alpha}}(x) \leq \delta$.

Otherwise, if $||x - x_r|| < 2^{-\sqrt{r} + O(\log r) + O(1)}$ for infinitely many $r \in \omega$, then for each such r,

$$K_{\sqrt{r}}^B(x') \leq O(\log r) + O(1) = O(\log \sqrt{r}),$$

implying $\dim^B(x) = 0$.

Finally, if neither condition holds, take any one of the infinitely many $r \in \omega$ for which

$$K_r(x, \Phi^{\boldsymbol{\alpha}}(x)) < \delta r + \frac{K_r(\boldsymbol{\alpha})}{2} - \frac{r}{n}.$$

Then, by Lemma 3.5.2, it must be that $K_r(x \mid \boldsymbol{\alpha}, x_r) < \delta \cdot r$, meaning $\dim^{B,\boldsymbol{\alpha}}(x) \leq \delta$. In any case, the Point-to-Set Principle 1.12.1 implies $\dim_{\mathrm{H}}(D_n^{\boldsymbol{\alpha}}(\delta)) \leq \delta$, as desired. $\square$

Chapter 4
Algorithmic Information Theory on Net Spaces

4.1 Background

As we have noted, algorithmic information theory has succeeded at characterizing certain fractal dimensions and outer measures on simple spaces in effective terms (e.g., the point-to-set principles 1.12.1 and 1.12.3). These simple spaces include Euclidean space as well as spaces of infinite sequences over finite alphabets. Yet, classical fractal dimensions such as Hausdorff dimension and packing dimension are well-defined over *arbitrary* metric spaces. So, it is natural to investigate over which settings AIT may successfully be applied to GMT.

E. Mayordomo isolated for her *nicely covered spaces* some conditions under which a metric space would admit a subset of the effective dimension equalities found in Theorem 1.9.2 [46]. Even more recently, J. Lutz, N. Lutz, and Mayordomo extended the Point-to-Set Principle to all separable metric spaces via an incompressibility approach [33]. They leave open the possibility for algorithmic dimension to be alternatively characterized in general metric spaces via effective mass distribution or betting strategies.

In this chapter, we develop the full algorithmic information theory framework for a broad class of metric spaces which comprises the two approaches known hitherto.

A standard result in geometric measure theory is one by A. Besicovitch claiming that within each compact subset of Euclidean space of positive $\mathcal{H}^s$-measure, one may find a subset whose $\mathcal{H}^s$-measure is in fact both non-zero and finite [3]. Besicovitch's proof involved constructing outer measures from premeasures defined on the collection of dyadic cubes over Euclidean space. Besicovitch further noticed that these outer measures would assign a comparable measure to arbitrary subsets as the s-dimensional Hausdorff outer measures $\mathcal{H}^s$.

D. Larman later generalized these notions to metric spaces covered by *nets* [27]: a net on a metric space may be described as a countable cover consisting of sufficiently nicely nested subsets; and a *net measure* is an outer measure produced from a given premeasure defined on a net. Call any metric space admitting a net to be a *net space*. C. Rogers and R. Davies succeeded at extending Besicovitch's result about compact sets to a broad class of net spaces using the properties of nets and net measures [55]. The essential properties of the collection of dyadic cubes on Euclidean space also play a similar role in the study of maximal functions in harmonic analysis. Evidently, net spaces should constitute fruitful settings over which to consider geometric measure theory.

In the language of nets, Euclidean space was shown to admit a net of dyadic cubes rich enough to simulate Hausdorff dimension only by covers comprising elements of that net. Note that in general, not all metric spaces admit nets, and not all metric spaces which do admit nets have a rich enough family of nets to simulate Hausdorff dimension.

Net spaces are particularly suited to the development of algorithmic information theory. The computability arguments made on either the nicely covered spaces of [46] or the separable metric spaces considered in [33] make use of countable collections of subsets with nice covering and nesting properties. Our approach will take advantage of the countability and tree-like structure of a net under set-inclusion to extend algorithmic information theory to net spaces. Whether the effective dimension notions produced in this extension will correspond to the classical Hausdorff dimension on that space will depend on the existence of nets with net measures comparable to the family $(\mathcal{H}^s)_{s\geq 0}$.

From this perspective, the standard Point-to-Set Principle 1.12.1 for Euclidean space may be viewed as the combination of two results: first, that Hausdorff dimension restricted to covers by dyadic cubes admits a point-to-set principle using effective covers also by dyadic cubes; and second, that the dyadic cubes net on Euclidean space admits comparable measures to each $\mathcal{H}^s$.

4.1.1 Meshes and Nets

Definition 4.1.1. We say that a collection $\mathcal{N}$ of non-empty subsets of a metric space Ω is a *mesh* on Ω if it satisfies the following properties:

(**M1**) $\mathcal{N}$ is countable;

(**M2**) Each element of $\mathcal{N}$ has at most finitely many supersets also in $\mathcal{N}$;

(**M3**) For any $x \in \Omega$ and $\varepsilon > 0$, x is contained in an element of $\mathcal{N}$ of diameter less than ε.

The pair $(\Omega, \mathcal{N})$ is then called a *mesh space*.

For any two mesh elements $N_1, N_2 \in \mathcal{N}$, let $N_1 \mid\mid N_2$ denote that N_1 and N_2 are *comparable* (under the containment relation), which means either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. Otherwise, N_1 and N_2 are said to be *incomparable*.

Definition 4.1.2. If $\mathcal{N}$ is a mesh on Ω, we call $\mathcal{N}$ a *net* on Ω if it further satisfies:

(**N4**) No element of $\mathcal{N}$ is contained in two incomparable elements from $\mathcal{N}$. That is, $N \subseteq N_1 \cap N_2$ implies $N_1 \mid\mid N_2$ for all $N, N_1, N_2 \in \mathcal{N}$.

The pair $(\Omega, \mathcal{N})$ is then called a *net space*.

This definition of a net differs from the definition provided by C. Rogers and R. Davies in [55] in the following ways. First, they ask that each element of a net be F_σ (that is, $\mathbf{\Sigma}^0_2$) in order to get useful results about the regularity of their net measures, as well as on how such net measures behave on compact sets with positive measure. For most of our purposes, we will not need a bound on the descriptive complexity of net elements. Second, (**N4**) slightly generalizes the incomparability axiom of Rogers and Davies which asks that any two non-trivially intersecting net elements be comparable. As we will see, many results from algorithmic information theory still hold under our weaker incomparability axiom (**N4**). We will call any net which satisfies the following stronger axiom to be a *layered-disjoint net*:

(**N4′**) No two incomparable elements of $\mathcal{N}$ may meet non-trivially. That is, $N_1 \cap N_2 \neq \emptyset$ implies $N_1 \mid\mid N_2$ for all $N_1, N_2 \in \mathcal{N}$.

The name is inspired by the *layered disjoint systems* of [6]. It will be clear that any net satisfying (**N4′**) induces a layered disjoint system in that sense. Any layered-disjoint net on Ω may also be viewed as a Lusin scheme on Ω in the sense of Definition 4.7.9 below. Whenever a layered-disjoint net comprises only F_σ sets, we will more specifically call it a *Rogers net*.

The elements of a mesh and net are arranged in a tree-like structure. To argue this, let us first note that the containment relation $\supseteq$ is *well-founded* on a mesh or net. That is, $\supseteq$ is a partial binary relation on $\mathcal{N} \times \mathcal{N}$ which always admits minimal elements, i.e.,

$$(\forall \mathcal{E} \subseteq \mathcal{N})\,[\mathcal{E} \neq \emptyset \implies (\exists N \in \mathcal{E})(\forall N' \in \mathcal{E})[N' \supseteq N \implies N' = N]]\,.$$

Since $\supseteq$ is both transitive and antisymmetric, the mesh axiom (**M2**) gives us that the containment relation $\supseteq$ is well-founded over any mesh.

In particular, we claim in the following proposition that the containment relation on a mesh induces a *directed, acyclic graph (DAG)* on the mesh elements. If $G = (V, E)$ is a graph, and $\to$ is a partial binary relation on $V \times V$ satisfying $x \to y \iff (x, y) \in E$, then G is a DAG if the transitive closure (i.e., the extended relation which relates two vertices so long as there is a finite directed path between them) of $\to$ is well-founded.

Proposition 4.1.3. *To any mesh $\mathcal{N}$ on a metric space Ω may be associated a directed, acyclic graph $\mathcal{G}$ whose vertices are the elements of $\mathcal{N}$, and whose directed edge relation ($\to$) is given by the immediate-predecessor relation:*

$$N_1 \to N_2 \iff N_2 \subsetneq N_1 \wedge (\forall N \in \mathcal{N})[N_2 \subsetneq N \subseteq N_1 \implies N = N_1].$$

Proof. By (**M2**), any set in $\mathcal{N}$ must have only finitely many supersets in $\mathcal{N}$. Therefore, all paths between vertices in $\mathcal{G}$ are of finite length. And by (**M1**), the degree of every vertex is countable, so $\mathcal{G}$ is a directed graph on $\mathcal{N}$. It remains to check that $\mathcal{G}$ is acyclic.

Let $\preceq_{\mathcal{N}}$ be the partial order on vertices of $\mathcal{G}$ induced by path-connectedness (i.e., the transitive closure of pred): define $N_1 \preceq_{\mathcal{N}} N_2$ if and only if there is a finite path in $\mathcal{G}$ from N_1 to N_2 (or, if $N_1 = N_2$). Then, $N_1 \preceq N_2$ holds if and only if $N_2 \subseteq N_1$. And as the containment relation is well-founded, it follows that $\mathcal{G}$ is directed acyclic. $\square$

By standard results on well-founded relations (e.g., Theorem 2.27 of [20]), one may assign to any mesh element $N \in \mathcal{N}$ its *rank*, which is given by the length of the longest path in $\mathcal{G}$ from a root element (i.e., an element having no strict superset in the mesh) of $\mathcal{N}$ to N. This may be defined inductively as follows: the rank of any root element in $\mathcal{N}$ is taken to be zero, and for any other $N \in \mathcal{N}$:

$$\operatorname{rk}(N) := 1 + \max\left\{\operatorname{rk}(N') : N \subsetneq N' \in \mathcal{N}\right\}.$$

For any mesh $\mathcal{N}$ and natural number $r \in \omega$, let $\mathcal{N}^{(r)} := \{N \in \mathcal{N} : \operatorname{rk}(N) = r\}$ denote the collection of elements in $\mathcal{N}$ with rank $r \in \omega$. Note that if $N \in \mathcal{N}^{(r)}$, then any superset of N in $\mathcal{N}$ must be of rank less than r.

It is well-known that on any DAG one may perform topological sort. This formalizes for meshes as follows.

Proposition 4.1.4. *Any mesh $\mathcal{N}$ admits a bijective indexing $\iota : \omega \to \mathcal{N}$ respecting the* *containment relation. That is, for any $i, j \in \omega$, we have*

$$\iota(i) \subseteq \iota(j) \implies i \geq j.$$

Proof. It suffices to find a bijection $\iota :\subseteq \omega \to \mathcal{N}$ respecting containment as in the statement. We take $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$ to denote a standard, bi-computable, injective pairing function monotonically-increasing in both its components.

Begin by enumerating each rank layer $\mathcal{N}^{(r)} = \left\{ N_0^{(r)}, N_1^{(r)} ..., \right\}$. We define ι inductively in the rank. For a given mesh element $N_e^{(0)} \in \mathcal{N}^{(0)}$, define the index of $N_e^{(0)}$ so that:

$$\iota : \langle 0, e \rangle \mapsto N_e^{(0)}.$$

For a fixed $r \in \omega$, suppose we have defined ι to surject onto all of $\mathcal{N}^{(r)}$, and take $N_e^{(r+1)} \in \mathcal{N}^{(r+1)}$. Assign an index to $N_e^{(r+1)}$ as follows:

$$I_e^{(r+1)} := \max_{f \in \omega} \left\{ \iota^{-1} \left(N_f^{(r)} \right) : N_e^{(r+1)} \subseteq N_f^{(r)} \right\};$$
$$\iota : \left\langle r + 1, \left\langle I_e^{(r+1)}, e \right\rangle \right\rangle \mapsto N_e^{(r+1)}.$$

First, notice that by induction, ι indeed surjects onto $\mathcal{N}$ since $(\mathcal{N}^{(r)})_r$ partitions $\mathcal{N}$. The indexing is injective since $\langle \cdot, \cdot \rangle$ is injective. It remains to check that ι respects containment.

Suppose $i, j \in \omega$ are such that $\iota(i) \subseteq \iota(j)$. It holds that either $\iota(i) = \iota(j)$ (and hence, $i = j$), or $\iota(i) \subsetneq \iota(j)$, meaning $\mathrm{rk}(\iota(j)) < \mathrm{rk}(\iota(i))$. Without loss of generality, we may assume that $\mathrm{rk}(\iota(j)) = \mathrm{rk}(\iota(i)) - 1$ (the general case would then follow by induction). Suppose $i = \left\langle r, \left\langle I_e^{(r)}, e \right\rangle \right\rangle$ for some $r > 0$ and $e \in \omega$. Then, by definition, $I_e^{(r)} \geq j$, giving $i > j$ by the properties of $\langle \cdot, \cdot \rangle$. □

Proposition 4.1.5. *The associated graph $\mathcal{G}$ to a net $\mathcal{N}$ is a forest of countably many countable trees. That is, the net $\mathcal{N}$ is embeddable into $\omega^{<\omega}$.*

Proof. Take $\mathcal{G}$ to be the directed, acyclic graph associated to $\mathcal{N}$ in Proposition 4.1.3. With the net axiom (**N4**), it follows that each element of $\mathcal{N}$ has at most one immediate predecessor with respect to containment That is, each vertex of $\mathcal{G}$ has an in-degree of at most 1. So $\mathcal{G}$ is graph-isomorphic to a countable collection of disjoint trees on $\omega^{<\omega}$. □

If $\mathcal{N}$ is a layered-disjoint net, then the sequence of rank layers $(\mathcal{N}^{(r)})_{r \in \omega}$ forms a layered disjoint system in the sense of [6]. Moreover, the infinite paths through the associated graph $\mathcal{G}$ which begin at indices of root elements (i.e., elements with no strict-superset in $\mathcal{N}$) are in a one-to-one correspondence with points in Ω.

Definition 4.1.6. For $x \in \Omega$ and mesh $\mathcal{N}$ on Ω with indexing ι, call any sequence $(i_n)_{n\in\omega} \subseteq \omega$ satisfying

$$x \in \bigcap_n \iota(i_n) \quad \text{and} \quad \operatorname{diam}(i_n) \to 0 \quad \text{as} \quad n \to \infty$$

an *$\mathcal{N}$-name of x*. Collect all $\mathcal{N}$-names of x in the set $\mathcal{R}(x)$.

Since incomparable mesh elements may overlap (i.e., have nontrivial intersection), each $\mathcal{R}(x)$ may be a large collection in general. But layered-disjoint nets admit unique names.

4.1.2 Computability for Meshes and Nets

Fix a mesh space $(\Omega, \mathcal{N})$, as well as an indexing $\iota : \omega \to \mathcal{N}$ respecting containment as in Proposition 4.1.4.

For the purposes of computability, we treat the mesh $\mathcal{N}$ as a structure with a corresponding *ω-presentation $\mathscr{R}$* to which we will assume to possess oracle access. We assume that $\mathscr{R}$ at least computes $\langle \text{in}, \text{pred}, \text{diam}, \text{root}\rangle$, where

- $\text{in} : \omega^2 \to \{\top, \bot\}$ is a relation on pairs of indices encoding the containment relation in the mesh; i.e., if $N_i, N_j \in \mathcal{N}$ satisfy $\iota(i) = N_i$ and $\iota(j) = N_j$, then

$$\text{in}(i, j) \iff N_i \subseteq N_j.$$

- $\text{pred} : \omega^2 \to \{\top, \bot\}$ is a relation on pairs of indices encoding the predecessor relation in the mesh; i.e., if $N_i, N_j \in \mathcal{N}$ satisfy $\iota(i) = N_i$ and $\iota(j) = N_j$, then

$$\text{pred}(i, j) \iff N_j \subsetneq N_i \wedge (\forall N \in \mathcal{N})[N_j \subsetneq N \subseteq N_i \implies N = N_i].$$

- $\text{diam} : \omega \to [0, \infty]$ is a map from indices to non-negative extended reals encoding the diameter function for net elements; i.e., if $\iota(i) = N_i \in \mathcal{N}$, then

$$\text{diam}(i) = \text{diam}_d(N_i) = \sup\{d(x, y) : x, y \in N_i\}.$$

- root : $\omega \to \{\top, \bot\}$ is the characteristic function for the set of indices of the root elements of the mesh; i.e., if $\iota(i) = N_i \in \mathcal{N}$, then

$$\text{root}(i) \iff (\forall N \in \mathcal{N})[N \supseteq N_i \implies N = N_i].$$

Note that in net spaces, the predecessor relation is equivalently characterized as:

$$\text{pred}(i,j) \iff N_j \subsetneq N_i \wedge (\forall N \in \mathcal{N})[N_j \subsetneq N \implies N_i \subseteq N].$$

Given a mesh space $(\Omega, \mathcal{N})$ and an ω-presentation $\mathscr{R}$ of $\mathcal{N}$ as described above, call the tuple $(\Omega, \mathcal{N}, \mathscr{R})$ a *represented mesh space.* If $\mathcal{N}$ is in fact a net, we might also call $(\Omega, \mathcal{N}, \mathscr{R})$ a *represented net space.*

Any computability notion considered over a mesh $\mathcal{N}$ must be taken with respect to some ω-presentation $\mathscr{R}$ of $\mathcal{N}$. An *effective mesh* is a mesh with a computable ω-presentation $\mathscr{R}$. We may drop the reference to a computable presentation, since having oracle access to it will not provide any extra computational power. Analogous definitions describe *effective nets* of each kind.

Recall the definition of a computable metric space from Section 1.4. We note that any separable metric space may be viewed as a computable metric space relative to some oracle which uniformly computes the pairwise distances between elements of the dense sequence.

Suppose the metric space (Ω, d) has an effective mesh $\mathcal{N}$ with a computable ω-presentation $\mathscr{R}$. And suppose there is a map $\chi : \omega \to \Omega$ such that for each $i \in \omega$, we have $\chi(i) \in \iota(i)$ and such that the map $i, j \mapsto d(\chi(i), \chi(j))$ is computable. Denote by $\alpha = (\alpha_i)_{i \in \omega}$ with $\alpha_i = \chi(i)$ the sequence built from the range of χ. Then, we call $(\Omega, d, \mathcal{N}, \mathscr{R}, \alpha)$ a *computable mesh space.* If $\mathcal{N}$ were in fact a net, we might also call this space a *computable net space.* And, without the mesh and its presentation, any computable mesh space is a computable metric space.

Proposition 4.1.7. *If $(\Omega, d, \mathcal{N}, \mathscr{R}, \alpha)$ is a computable mesh space, then (Ω, d, α) is a computable metric space.*

Proof. We already have that the pairwise distances between elements of α are computable by the definition of a computable mesh space. It remains to show that this sequence is dense in Ω. Take $x \in \Omega$ and $\delta > 0$. By the mesh axiom (**M3**), there exists a mesh element $N \in \mathcal{N}$ such that $x \in N$ and $\text{diam}_d(N) < \delta$. By the triangle inequality, $N \subseteq B_\delta(x)$. But if $\iota(i) = N$ for some $i \in \omega$, then $\chi(i) \in N$ implies that $\chi(i) \in B_\delta(x)$ as well. So $(\chi(i))_i$ is dense in Ω. □

Note that many of the basic examples of mesh or net spaces are also computable metric spaces. However, the notion of a space with an effective mesh is not generally comparable with the notion of a computable metric space.

4.1.3 Examples

Let us begin by extending an example from [46].

Example 4.1.8. For $m \in \omega$, m-dimensional Euclidean space $\mathbb{R}^m$–as a metric space whose metric is based on a Borel measure ν on $\mathbb{R}^m$–is a net space when paired with the net $\mathcal{Q}^m$ composed of all half-open, dyadic rational cubes (as defined in Section 1.2). In fact, $\mathcal{Q}^m$ is a Rogers net on $\mathbb{R}^m$. We have (**M3**) since for each x there exists a sequence $(Q_j)_j \subset \mathcal{Q}^m$ such that $\bigcap_j Q_j = \{x\}$ and ν satisfies continuity from above. A simple dove-tailing algorithm can index $\mathcal{Q}^m$ while respecting the containment relation. Under this enumeration $(Q_i)_{i\in\omega}$, it is also straightforward to compute the containment relation, predecessor relation, and indices of any root elements (those cubes with $n = 0$).

In particular, $\mathbb{R}^m$ with the usual Euclidean metric has an effective Rogers net. Notice that $\operatorname{diam}(Q(\mathbf{z}, \mathbf{a}, n)) = \sqrt{m} \cdot 2^{-n}$ is a computable function. The centers of all cubes in $\mathcal{Q}^m$ constitute a dense subset of $\mathbb{R}^m$ with computable pairwise distances. Therefore, $\mathbb{R}^m$ under the Euclidean metric and dyadic cubes net make a computable net space.

Another example comes from [55].

Definition 4.1.9. A metric space (Ω, d) is said to be an *ultrametric space* if, whenever $x, y, z \in \Omega$, then

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Example 4.1.10. Let $(\Omega,\, d)$ be a separable ultrametric space. Then Ω has many layered-disjoint nets. Fix a strictly-decreasing sequence of positive reals $(d_i)_{i\in\omega}$ converging to zero. And fix a countable, dense sequence $(x_j)_{j\in\omega}$ in Ω. For any $x \in \Omega$ and $i \in \omega$, denote:

$$N(x, i) := \{y \in \Omega : d(y, x) \leq d_i\}.$$

We claim that the collection $\mathcal{N} := \{N(x_j, i) : i, j \in \omega\}$ is a net on Ω. Since Ω is an ultrametric space, each $N(x, i)$ is open and non-empty containing x. Again, Ω being an ultrametric space gives us the following fact: if ever we have for some $x, y \in \Omega$ and $0 \leq i \leq j$ that $N(x,i) \cap N(y,j) \neq \emptyset$, it follows that that exists z in their intersection satisfying,

$$d(x,z) \leq d_i \text{ and } d(y,z) \leq d_j \implies d(x,y) \leq \max\{d_i, d_j\} = d_i.$$

Thus, for any $w \in N(y,j)$, both $d(x,y) \leq d_i$ and $d(y,w) \leq d_i$, giving $d(x,w) \leq d_i$ as well. Thus, $w \in N(x,i)$. This means that any nontrivial overlap between two elements implies the elements are comparable under containment, giving (**N4′**). Since $\operatorname{diam}_d(N(x,i)) \leq d_i$, the mesh axiom (**M3**) follows. Finally, if $x \in \Omega$ and $i \in \omega$, we consider which $N(y,j)$ might contain $N(x,i)$. And since $(d_i)_i$ is strictly-decreasing, there must exist some smallest $k+1 \in \omega$ so that $N(x,k+1)$ is a proper subset of $N(x,i)$. By (**N4′**), any $N(y,j) \supseteq N(x,i)$ must then match one of the finitely many elements $N(x,1)$, $N(x,2)$, ..., $N(x,k)$. This gives (**M2**). So $\mathcal{N}$ is indeed a layered-disjoint net on Ω.

Example 4.1.11. Take Baire space ω^ω, equipped with the product topology from the discrete topology on ω and the compatible metric:

$$d(\alpha,\beta) := \begin{cases} 0 & \alpha = \beta, \\ 2^{-\min\{i : \alpha(i) \neq \beta(i)\}} & \alpha \neq \beta, \end{cases} \qquad \alpha, \beta \in \omega^\omega.$$

First, notice that d is in fact an ultrametric under which Baire space is separable and has no isolated points. By Example 4.1.10, this ultrametric space has many layered-disjoint nets. We may find an effective one explicitly.

If $I \subset \omega$ is a finite set of indices with corresponding tuple of fixed values $\mathbf{x}_I = (x_i \in \omega : i \in I)$, then the *cylinder set above* $\mathbf{x}_I$ is defined as the set:

$$C[\mathbf{x}_I] := \{\alpha \in \omega^\omega : \alpha(i) = x_i \text{ for all } i \in I\}.$$

Denote the countable collection of all such cylinder sets by $\mathcal{C}$: the usual collection of basic open sets which generates the product topology on Baire space. We see that $\mathcal{C}$ forms a mesh over Baire space. For any $n \in \omega$ and $\alpha \in \omega^\omega$, it holds that $\alpha \in C[\alpha \restriction n]$ and $\bigcap_n C[\alpha \restriction n] = \{\alpha\}$, which forms an infinite, nesting sequence of net elements with diameters shrinking to zero under d. This gives (**M3**). But $\mathcal{C}$ is not a net on Baire space. The two incomparable cylinders $C[\alpha_0 = 0]$ and $C[\alpha_1 = 0]$ constitute a minimal counterexample to (**N4**): their intersection is yet another cylinder of the form $C[\alpha \restriction 2 = (0,0)]$. Instead, we take the sub-collection $\mathcal{N} \subset \mathcal{C}$ comprising only those cylinders above tuples defined on initial segments of ω: i.e., $\mathcal{N} := \{C[\sigma] : \sigma \in \omega^{<\omega}\}$. In fact, this collection $\mathcal{N}$ generates the same product topology as $\mathcal{C}$ on Baire space, and further qualifies as a net.

Since any non-trivial intersection between two cylinders from $\mathcal{N}$ implies their comparability, we have (**N4′**). Notice too that the chain $(C[\alpha \upharpoonright n])_{n\in\omega} \subset \mathcal{C}$ we previously identified to represent a given sequence α only employs cylinders from $\mathcal{N}$. It is computable to obtain an enumeration of $\mathcal{N}$ respecting containment, and this enumeration can compute the containment relation, predecessor relation, and indices of any root elements (i.e., $C[()]$). Notice too that the diameter function on $\mathcal{N}$ is computable: for any $\sigma \in \omega^{<\omega}$,

$$\operatorname{diam}_d(C[\sigma]) = 2^{-\operatorname{len}(\sigma)}.$$

So, the net $\mathcal{N}$ has a computable ω-presentation $\mathscr{R}$, meaning $\mathcal{N}$ is an effective Rogers net on (ω^ω, d). Moreover, selecting from each $C[\sigma]$ the point $\sigma^\frown 0^\omega$, one may effectively compute their pairwise distances using the enumeration of $\mathcal{N}$. So $(\omega^\omega, d, \mathcal{N}, \mathscr{R}, (\sigma^\frown 0^\omega)_\sigma)$ constitutes a computable net space.

Example 4.1.12. Consider the field of p-adic numbers $\mathbb{Q}_p$ where $p \in \omega$ is prime, along with the standard p-adic norm $|\cdot|_p$. The p-adic norm actually induces an ultrametric on $\mathbb{Q}_p$ with computable dense subset $\mathbb{Q}$. By Example 4.1.10, this ultrametric space has many layered-disjoint nets.

If I is an initial segment of $\mathbb{Z}$ with corresponding tuple $\mathbf{a}_I = (a_i \in \{0, ..., p-1\} : i \in I)$ of fixed values–only finitely many of which may be non-zero–then the *cylinder set above* $\mathbf{a}_I$ is defined as the set:

$$C[\mathbf{a}_I] := \left\{ x \in \mathbb{Q}_p : x = \sum_{i\in\mathbb{Z}} x_i p^i, \quad \text{where} \quad x_i \in \{0, ..., p-1\} \text{ and } x_i = a_i \text{ for all } i \in I \right\}.$$

We only permit tuples $\mathbf{a}_I$ of all zero values if I is an initial segment containing 0. Denote the countable collection of all cylinder sets above permitted tuples by $\mathcal{C}$: a collection of basic open sets which generates the product topology on $\mathbb{Q}_p$ (being the product of the topology on $\mathbb{Z}_p$, which is itself a product topology from the discrete topology on $\{0, ..., p-1\}$). We claim that $\mathcal{C}$ is an effective Rogers net on $\mathbb{Q}_p$.

Clearly, $\mathcal{C}$ is countable. A superset in $\mathcal{C}$ of a cylinder $C[\mathbf{a}_I] \in \mathcal{C}$ must be a cylinder above the nonzero restriction of $\mathbf{a}_I$ to some initial segment of I. There are only finitely many such restrictions, so we have (**M2**). For any $C[\mathbf{a}_I]$, if the restriction $C[\mathbf{a}_{I\setminus\{\max(I)\}}]$ of $\mathbf{a}_I$ to the next shortest initial segment is a cylinder in $\mathcal{C}$, then $C[\mathbf{a}_{I\setminus\{\max(I)\}}]$ is the immediate predecessor of $C[\mathbf{a}_I]$. Otherwise, $C[\mathbf{a}_I]$ has no immediate predecessor. It is straightforward to check that any two overlapping cylinders from $\mathcal{C}$ must be above tuples which agree on a common initial segment of ω, so are necessarily comparable. This gives (**N4′**). Now, if $x \in \mathbb{Q}_p$, then x is uniquely expressible in the form $p^k \cdot u$ for some $k \in \mathbb{Z}$ and invertible $u \in \mathbb{Z}_p^\times$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Then $(C\,[x[k, ..., k+n-1]])_{n\in\omega}$ is an infinite, nested chain of cylinders from $\mathcal{C}$ each containing x and shrinking in diameter to zero, since,

$$\mathrm{diam}_p(C\,[x[k, ..., k+n-1]]) = p^{-(k+n)} \to 0 \quad \text{as} \quad n \to \infty.$$

This gives (**M3**). We may index these cylinders by ω as follows: lexicographically order all of the tuples whose earliest non-zero value occurs at a common index k, where we consider the symbols to be ordered according to $0 < 1 < \cdots < p-1 <$ `blank`; then interleave those orders across all $k \in \mathbb{Z}$ in the fashion $k = 0, +1, -1, +2, -2, ...$. From this effective enumeration of $\mathcal{C}$, one may compute the containment relation, immediate predecessor relation, and the indices of all root elements (i.e., those tuples which are only non-zero on the last index of the initial segment on which they are defined). Moreover, the diameter function is computable from the index of the cylinder, so $\mathcal{C}$ is indeed an effective Rogers net on $\mathbb{Q}_p$. Finally, selecting the point $x = \sum_{i\in I} a_i p^i$ from $C[\mathbf{a}_I]$ provides a dense subset with computable pairwise distances. This dense subset and effective net $\mathcal{C}$ make $\mathbb{Q}_p$ a computable net space.

4.1.4 Comparison to Nicely Covered Metric Spaces

We may view the development of algorithmic information theory over net spaces as a generalization of the work done in [46] for those metric spaces with computable nice covers. Let us recall the definition of a computably nicely covered metric space.

Definition 4.1.13. Let (Ω, d) be a metric space without isolated points. A *nice cover* of Ω is a sequence $(\mathcal{B}_n)_n$ with $\mathcal{B}_n \subseteq \mathcal{P}(\Omega)$ for each $n \in \omega$ satisfying the axioms:

(**A1**) For each $n \in \omega$ and each $U \in B_n$, we have $|\{V \in \mathcal{B}_{n+1} : V \subseteq U\}| < \infty$.

(**A2**) For each $n \in \omega$, each $U \in B_n$, and each $m < n$, there exists a unique $V \in \mathcal{B}_m$ such that $U \subseteq V$.

(**A3**) For each $n \in \omega$, we have $\inf \{\mathrm{diam}_d(U) : U \in \mathcal{B}_n\} > 0$.

(**A4**) There exists $c \in \omega$ such that for each $A \subseteq \Omega$ with $0 < \mathrm{diam}_d(A) < 1$, there exists $\{U_1, ..., U_c\} \subseteq \bigcup_n \mathcal{B}_n$ such that,

$$A \subseteq \bigcup_{i=1}^{c} U_i \quad \text{and} \quad (\forall i \in \omega)[\mathrm{diam}_d(U_i) < c \cdot \mathrm{diam}_d(A)].$$

Furthermore, if $(\mathcal{B}_n)_n$ is a nice cover of Ω, Σ is a finite alphabet, and $\delta : \Sigma^{<\omega} \to \bigcup_n \mathcal{B}_n$ is a bijection, then it is said that $(\Omega, (\mathcal{B}_n)_n, \delta)$ *has a computable nice cover* if both:

(**B1**) $\mathrm{diam}_d \circ \delta$ is a computable function, and

(**B2**) The function succ : $\Sigma^{<\omega} \times \omega \to \Sigma^{<\omega}$ on pairs (w, n) satisfying $\delta(w) \in \mathcal{B}_n$, defined by $\mathrm{succ}(w, n) = \langle w_1, ..., w_k \rangle$ and $\{V \in \mathcal{B}_{n+1} : V \subseteq \delta(w)\} = \{\delta(w_1), ..., \delta(w_k)\}$ is computable. That is, there is a computable map giving the indices of immediate successors of the cover element with the given index w.

Proposition 4.1.14. *If (Ω, δ) has a computable nice cover $(\mathcal{B}_n)_n$, then $\mathcal{N} := \bigcup_n \mathcal{B}_n$ is a layered-disjoint net on Ω with an ω-presentation $\mathscr{R}$ computable in β' (the Turing jump of β), where $\beta : \omega \to \Sigma^{<\omega}$ is some bijection such that $\delta \circ \beta : \omega \to \mathcal{N}$ is an indexing of $\mathcal{N}$ respecting set-inclusion.*

Proof. We first check that the mesh and net axioms are met.

(**M1**) δ is a bijection between $\Sigma^{<\omega}$ and $\mathcal{N}$. So $\mathcal{N}$ is countable.

(**M2**) By induction on the layer n, (**A2**) implies that the the number of supersets of an element of $\mathcal{N}$ is no greater than the layer at which the element lies.

(**M3**) Let $x \in \Omega$ and fix $0 < \varepsilon < 1$. Since x is not isolated, there exists a subset $x \in A \subseteq \Omega$ satisfying $0 < \mathrm{diam}_d(A) < \varepsilon$. Thus, by (**A4**), there exists $\{U_1, ..., U_c\} \subseteq \mathcal{N}$ covering A and satisfying $\mathrm{diam}_d(U_i) < c \cdot \mathrm{diam}_d(A)$ for each $1 \leq i \leq c$. Let i be an index for which $x \in U_i$. Then,

$$\mathrm{diam}_d(U_i) < c \cdot \mathrm{diam}_d(A) < c \cdot \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we may indeed find cover elements of arbitrarily small diameter covering x.

(**N4′**) By induction on the layer n, no element of $\mathcal{N}$ may have two incomparable supersets from the cover $\mathcal{N}$ without violating (**A2**).

Now, denote $\iota := \delta \circ \beta$. Then we may compute an ω-presentation $\mathscr{R}$ of $\mathcal{N}$ as follows:

- By (**B2**), $\mathrm{in} : \omega^2 \to \{\top, \bot\}$ may be defined by the Σ_1-formula in γ: that $\mathrm{in}(i,j)$ holds if and only if:

$$(\exists n, C \in \omega) \left[\begin{array}{l} (C \text{ is the code for the tuple } (i_1, ..., i_n) \in \omega^n) \\ \wedge (\beta(j) \in \mathrm{succ}(\beta(i), \mathrm{len}(\beta(i)))) \wedge \cdots \wedge (\beta(i_n) \in \mathrm{succ}(\beta(j), \mathrm{len}(\beta(j)))) \end{array} \right].$$

- By (**B2**), $\mathrm{pred} : \omega^2 \to \{\top, \bot\}$ may be defined computably in β by the formula:

$$\mathrm{pred}(i,j) :\iff \beta(j) \in \mathrm{succ}(\beta(j), \mathrm{len}(\beta(j)))$$

- By (**B1**), $\mathrm{diam} : \omega \to [0, \infty]$ may be defined computably in β by the formula:

$$\mathrm{diam}(i) := (\mathrm{diam}_d \circ \delta)(\beta(i)).$$

- By (**B2**), $\mathrm{root} : \omega \to \{\top, \bot\}$ may be defined by a Π_1-formula in β:

$$\mathrm{root}(i) :\iff (\forall w \in \Sigma^{<\omega}) \left[\beta(i) \notin \mathrm{succ}(w, \mathrm{len}(w))\right].$$

Therefore, $\mathscr{R} = \langle \mathrm{in}, \mathrm{pred}, \mathrm{diam}, \mathrm{root} \rangle$ is an ω-presentation of $\mathcal{N}$ computable from β', as desired. Notice that the nice cover axioms (**A1**) and (**A3**) were not utilized in the course of this proof. □

4.2 Net Measures

For this section, fix a metric space (Ω, d).

4.2.1 Net Measures

Recall the essential definitions of geometric measure theory from Section 1.3. In particular, we continue to use Method II for building outer measures from premeasures.

Definition 4.2.1. Let $\mathcal{N}$ be a net on Ω be a collection of subsets, and ρ a premeasure on $\mathcal{N} \cup \{\emptyset\}$. We call ρ a *net premeasure* for $\mathcal{N}$. Then for each $\delta > 0$, the *ρ-dimensional δ-content* under Method II is denoted by $\mathcal{H}^\rho_\delta$, while $\mathcal{H}^\rho$ denotes the corresponding *net (outer) measure* associated to ρ under Method II.

It will serve us to establish a notion of *comparability* between premeasures.

Definition 4.2.2. If ν and ρ are both premeasures on Ω, we say that ν is *comparable* to ρ (denoted $\nu \asymp \rho$) if for all $X \subseteq \Omega$,

$$\mathcal{H}^{\nu}(X) = 0 \iff \mathcal{H}^{\rho}(X) = 0.$$

Lemma 4.2.3. *Let ν and ρ both be premeasures on Ω. Suppose for each $X \subseteq \Omega$, there exist functions f and g satisfying:*

- *$f : [0, +\infty] \to [0, +\infty]$ is continuous on the right on $[0, +\infty)$ and $f(0) = 0$,*
- *$g : (0, +\infty) \to (0, +\infty)$ satisfies $g(\delta) \to 0^+$ as $\delta \to 0^+$,*

such that for all $\delta > 0$,

$$\mathcal{H}^{\nu}_{\delta}(X) \leq \mathcal{H}^{\rho}_{\delta}(X) \leq f\left(\mathcal{H}^{\nu}_{g(\delta)}(X)\right). \tag{4.1}$$

Then ν and ρ are comparable.

Proof. It is clear from (4.1) that $\mathcal{H}^{\rho}(X) = 0$ implies $\mathcal{H}^{\nu}(X) = 0$ for all $X \subseteq \Omega$. For the reverse direction, assume $\mathcal{H}^{\nu}(X) = 0$. Then by (4.1) and the assumptions on f and g,

$$\mathcal{H}^{\rho}(X) = \lim_{\delta \to 0^+} \mathcal{H}^{\rho}_{\delta}(X) \leq \lim_{\delta \to 0^+} f\left(\mathcal{H}^{\nu}_{g(\delta)}(X)\right) = f\left(\lim_{\delta \to 0^+} \mathcal{H}^{\nu}_{g(\delta)}(X)\right) = f(\mathcal{H}^{\nu}(X)) = 0.$$

□

We will also make use of a stronger form of comparability which we call *commensurate up to multiplicative constants.*

Definition 4.2.4. If ν and ρ are both premeasures on Ω, we say that ν and ρ are *commensurate up to multiplicative constants* (denoted $\rho = \Theta(\nu)$) if there exist constants $0 < c \leq C < +\infty$ such that for all $X \subseteq \Omega$,

$$c \cdot \mathcal{H}^{\nu}(X) \leq \mathcal{H}^{\rho}(X) \leq C \cdot \mathcal{H}^{\nu}(X).$$

4.2.2 Net Dimension

Let us now introduce a notion of Hausdorff dimension restricted to a net.

Definition 4.2.5. For any $X \subseteq \Omega$, denote by $\dim_{\mathcal{N}}(X)$ the *Hausdorff dimension of X restricted to the net* $\mathcal{N}$, which is defined as:

$$\dim_{\mathcal{N}}(X) := \inf \{s \geq 0 : (\exists \rho \asymp \rho_s) [X \text{ is } \rho\text{-null for the net premeasure } \rho \text{ on } \mathcal{N}]\},$$

or $+\infty$ when the infimum is taken over the empty set.

Equivalently, we may write this quantity in terms of the dimension functions:

$$\dim_{\mathcal{N}}(X) = \inf \left\{ s \geq 0 : \mathcal{H}^{\rho_s \restriction \mathcal{N}}(X) = 0 \right\},$$

where, for any premeasure ρ on Ω, the notation $\rho \restriction \mathcal{N}$ denotes the premeasure obtained by restricting ρ to $\mathcal{N} \cup \{\emptyset\}$:

$$(\rho \restriction \mathcal{N})(X) = \begin{cases} \rho(X) & X \in \mathcal{N}, \\ 0 & X = \emptyset, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Unpacking the definitions, we may write Hausdorff dimension restricted to a net even more explicitly:

$$\dim_{\mathcal{N}}(X) = \inf \{s \geq 0 : (\mathcal{H} \restriction \mathcal{N})^s(X) = 0\},$$

where we may perform Method II on any premeasure ρ by covers consisting only of elements from an arbitrary collection $\mathcal{N} \subseteq \mathcal{P}(\Omega)$:

$$(\mathcal{H} \restriction \mathcal{N})^\rho(X) := \sup_{\delta > 0} (\mathcal{H} \restriction \mathcal{N})^\rho_\delta(X), \quad \text{where}$$
$$(\mathcal{H} \restriction \mathcal{N})^\rho_\delta(X) := \inf_{(C_i)_{i \in \omega}} \left\{ \sum_i \rho(C_i) : C_i \in \mathcal{N}, \operatorname{diam}_d(C_i) \leq \delta, \bigcup_i C_i \supseteq X \right\}.$$

For the same reasons as in the unrestricted case, this quantity is well-defined. Note that as $\rho_s \restriction \mathcal{N}$ is a net premeasure on $\mathcal{N}$, the computation of Hausdorff dimension restricted to $\mathcal{N}$ permits only those covers whose cover elements are sourced from the net $\mathcal{N}$. In the case of dyadic cubes $\mathcal{N} = \mathcal{Q}$ on Euclidean space, $(\mathcal{H} \restriction \mathcal{Q})^s_\delta$ matches K. Falconer's $\mathscr{M}^s_\delta$ in Chapter 5 of [13].

Note that one may also perform Method I restricted to an arbitrary collection

$\mathcal{N} \subseteq \mathcal{P}(\Omega)$:

$$(\mathcal{H} \upharpoonright \mathcal{N})^\rho_{\mathrm{I}}(X) := \inf_{(C_i)_{i\in\omega}} \left\{ \sum_i \rho(C_i) : C_i \in \mathcal{N}, \bigcup_i C_i \supseteq X \right\}.$$

We may now introduce net dimension.

Definition 4.2.6. For any $X \subseteq \Omega$, define the *net dimension of* X to be:

$$\dim_{\mathrm{net}}(X) := \inf \left\{ \dim_{\mathcal{N}}(X) : \mathcal{N} \text{ is a net on } \Omega \right\},$$

or $+\infty$ when the infimum is taken over the empty set.

Net dimension captures the ability to evaluate the Hausdorff dimension of a subset only through the various nets that might exist on the underlying metric space.

4.2.3 Basic Facts about Net Measures

A few results about net measures come directly from Rogers [55], including the *Increasing Sets Lemma* and a stability result. Both of these results apply to layered-disjoint nets, which satisfy the stronger incomparability axiom (**N4′**).

Theorem 4.2.7 (Increasing Sets Lemma, Theorem 52 of [55])**.** *Let $\mathcal{N}$ be a layered-disjoint net on a metric space (Ω, d), and ρ be a net premeasure for $\mathcal{N}$, and $\delta > 0$. Then for any increasing sequence $(X_n)_{n\in\omega}$ of subsets of Ω, we have $\mu\left(\bigcup_n X_n\right) = \sup_n \mu(X_n)$.*

Theorem 4.2.8 (Stability, Theorem 53 of [55])**.** *Let $\mathcal{N}$ be a layered-disjoint net on a metric space (Ω, d) and ρ be a net premeasure for $\mathcal{N}$. Define for each $n \in \omega$ the sub-net $\mathcal{N}^+_n$ by removing from $\mathcal{N}$ the first n many elements of positive diameter under the indexing ι. Then $\mathcal{H}^\rho_{\mathrm{I}}(X) = \sup_n (\mathcal{H} \upharpoonright \mathcal{N}^+_n)^\rho_{\mathrm{I}} = \sup_n \mathcal{H}^{\rho \upharpoonright \mathcal{N}^+_n}_{\mathrm{I}}$.*

4.3 Net Semimeasures

Recall that in Section 1.8, we discussed two approaches towards mass spreading along the cylinder sets over Cantor space, each of which admitted optimal elements and corresponding complexity notions for points in Cantor space. We may extend these approaches to maps which spread mass across the elements of a mesh, obtaining analogous results about the existence of optimal elements and corresponding complexity notions. As in Cantor space, a *discrete* assessment would not distinguish between elements of the mesh, whereas a *continuous* assessment would respect the DAG structure between mesh elements encoded in the containment relation. We begin by defining a continuous mesh semimeasure on a represented mesh space. For the rest of the section, fix a metric space (Ω, d) and a represented mesh space $(\Omega, \mathcal{N}, \mathscr{R})$.

4.3.1 Continuous Mesh Semimeasures

Definition 4.3.1. A function $M : \omega \to [0,1]$ is a *continuous mesh semimeasure on* $(\Omega, \mathcal{N})$ if it satisfies both:

$$M(i) \geq \sum_{\text{pred}(i,j)} M(j), \quad \text{and} \tag{4.2}$$

$$1 \geq \sum_{\text{root}(i)} M(i), \tag{4.3}$$

where root is the predicate encoding the root elements of $\mathcal{N}$.

In words, the *continuous semimeasure condition* in (4.2) asks that the map spread measure in such a way so that the measure assigned to a fixed mesh element is never exceeded by the sum of the measure assigned to its immediate successors. The boundedness condition in (4.3) asks that the map begin with no more than unity amount of measure across all of the mesh's root elements. Note that the maps $M : i \mapsto 0$ for all $i \in \omega$ trivially qualifies as a continuous mesh semimeasure.

For a fixed mesh space $(\Omega, \mathcal{N})$ and indexing ι, let $\mathcal{E} \sqsubseteq_{\text{D}} i$ denote

$$[j \in \mathcal{E} \implies \iota(j) \subseteq \iota(i)] \wedge [j, k \in \mathcal{E} \wedge j \neq k \implies \iota(j) \cap \iota(k) = \emptyset],$$

meaning $\mathcal{E}$ is a collection of indices of mutually-disjoint mesh elements which are all subsets of the i-th mesh element. If $(\Omega, \mathcal{N}, \mathscr{R})$ is in fact a represented net space, let $\mathcal{E} \sqsubseteq_{\text{PF}} i$ denote that $\mathcal{E}$ contains the indices of a collection of net elements whose addresses in the associated forest $\mathcal{G}$ form a prefix-free collection all extending the address of the i-th net element. Of course, $\mathcal{E} \sqsubseteq_{\text{D}} i$ implies $\mathcal{E} \sqsubseteq_{\text{PF}} i$ over net spaces, and the two are equivalent over layered-disjoint nets.

A straightforward argument by induction gives the following result about how continuous mesh semimeasures spread measure across incomparable sequences of mesh elements. Compare this to Property 3.8 in [46].

Proposition 4.3.2 (Generalized Kraft Inequality). *Let M be a continuous mesh semimeasure. If $\mathcal{E} \sqsubseteq_{\mathrm{D}} i$, then*

$$M(i) \geq \sum_{e \in \mathcal{E}} M(e).$$

Moreover, we have an alternate characterization of *continuous net semimeasures* (i.e., the continuous mesh semimeasures over a net).

Proposition 4.3.3. *M is a continuous net semimeasure if and only if both M satisfies (4.3) and, for all $i \in \omega$ and $\mathcal{E} \sqsubseteq_{\mathrm{PF}} i$,*

$$M(i) \geq \sum_{e \in \mathcal{E}} M(e).$$

Recall the discussion from Section 1.4 on lower-semicomputability. A continuous mesh semimeasure M on $(\Omega, \mathcal{N})$ is *lower-semicomputable* if its lower-graph is computably enumerable relative to the given ω-presentation $\mathscr{R}$. Collect into $\mathcal{M}$ all lower-semicomputable continuous mesh semimeasures in $\mathcal{N}$ with respect to the given ω-presentation $\mathscr{R}$. Then, as in the case of Cantor space, a continuous semimeasure from $\mathcal{M}$ is *optimal* if it multiplicatively dominates all other semimeasures from $\mathcal{M}$.

Theorem 4.3.4. *Over a given represented mesh space $(\Omega, \mathcal{N}, \mathscr{R})$, there exists an optimal lower-semicomputable continuous mesh semimeasure.*

Our proof will loosely follow the structure outlined in M. Li and P. Vitányi's Theorem 4.5.1 for constructing an optimal lower-semicomputable continuous semimeasure over some finitely-branching sequence space [29]. We will use that any element of $\mathcal{M}$ is $[0, 1]$-valued, which follows from the second continuous mesh semimeasure property (4.3).

Proof. We build our optimal element using left-approximators to the continuous semimeasures in $\mathcal{M}$. In the first step of the construction, we obtain an enumeration of left-approximators $\widehat{f}$ to the non-negative, lower-semicomputable maps $f \geq 0$. Then, in the next step, each $\widehat{f}$ is turned into a left-approximator for some lower-semicomputable continuous mesh semimeasure. Finally, we take a weighted sum of all these left-approximators to the elements of $\mathcal{M}$; this defines a computable map which left-approximates an optimal element of $\mathcal{M}$.

First, note that it is possible to enumerate all $\mathscr{R}$-computable functions of the form $\phi : \omega^2 \to \mathbb{Q} \cap [0, +\infty)$. For any lower-semicomputable map $f : \omega \to [0, \infty)$, the discussion in Section 1.4, guarantees that at least one of the ϕ above will left-approximate f.

Next, use a given ϕ to define another map $\psi : \omega^2 \to \mathbb{R}$ meant to be a left-approximator to some semimeasure in $\mathcal{M}$. By dovetailing this staged process across the enumeration of all ϕ, we will have described a uniform algorithm for approximating all elements of $\mathcal{M}$ from below (computably in $\mathscr{R}$), exactly what is needed for building an optimal element of $\mathcal{M}$.

```
Stage r = 0:
  Initialize ψ(i,0) = 0 for all i ∈ ω;
Stage r > 0:
  Initialize the temporary variable ψ̃_r(i) = 0 for all i ∈ ω;
  For i := r − 1, r − 2, ..., 0 do:
    Set ψ̃_r(i) = max{φ(i,r), Σ_{j<r} {ψ̃_r(j) : pred(i,j)}};
  If Σ{ψ̃_r(k) : root(k) ∧ k < r} > 1:
    Set ψ(i,r) := ψ(i,r − 1) for all i ∈ ω;
  Else:
    Set ψ(i,r) := ψ̃_r(i) for all i ∈ ω;
```

In words, we use $\phi(\cdot, r)$ to build a semimeasure $\widetilde{\psi}_r$ which only assigns non-zero values to indices below r. So long as the proposed $\widetilde{\psi}_r$ does not assign more than unit mass to the roots of the mesh, we use $\widetilde{\psi}_r$ to update $\psi(\cdot, r)$. Otherwise, we carry over the values of $\psi(\cdot, r-1)$ from the previous stage.

If at any stage the construction attempts to assign more than unit mass across the root elements, the resulting ψ will not be updated beyond that stage. Such a map will trivially satisfy $\psi(\cdot, r) : \omega \to \mathbb{R}$ being a continuous mesh semimeasure for each $r \in \omega$.

Otherwise, the map ψ is updated successfully at each stage. We have that the limiting map $M_\phi : i \mapsto \lim_{r\to\infty} \psi(i, r)$ is a continuous mesh semimeasure on $\mathcal{N}$. The second condition (4.3) is clear, so it remains to demonstrate the first property (4.2).

Fix $i \in \omega$. Should i have only finitely many immediate successors j, then by stage $r > \max\{j : \text{pred}(i, j)\}$, it is guaranteed by construction that $\widetilde{\psi}_r(i) \geq \sum_{\text{pred}(i,j)} \widetilde{\psi}_r(j)$. So, in the limit,

$$\psi(i, r) \geq \sum_{\text{pred}(i,j)} \psi(j, r).$$

Otherwise, i has infinitely many immediate successors. Note that by assumption, $\sum_{\text{pred}(i,j)} \psi(j) \leq 1$. In particular, this sum converges. Fix $\varepsilon > 0$. There exists a sufficiently large integer $J \gg 1$ so that the partial sum satisfies:

$$\sum_{\substack{\mathrm{pred}(i,j) \\ j<J}} \psi(j,r) \geq \sum_{\mathrm{pred}(i,j)} \psi(j,r) - \varepsilon.$$

And for any stage $r > J$, it is guaranteed by construction that

$$\widetilde{\psi}_r(i) \geq \sum_{\substack{\mathrm{i,j} \\ j<J}} \widetilde{\psi}_r(j) \implies \psi(i,r) \geq \sum_{\substack{\mathrm{i,j} \\ j<J}} \psi(j,r) \geq \sum_{\mathrm{pred}(i,j)} \psi(j,r) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have exhibited the continuous mesh semimeasure property.

Thus, no matter ϕ, the limiting map $M_\phi : i \mapsto \lim_{r\to\infty} \psi(i,r)$ from this construction will always satisfy $M_\phi \in \mathcal{M}$.

Furthermore, for a given $M \in \mathcal{M}$, there exists an $\mathscr{R}$-computable left-approximator ϕ to M where each map $i \mapsto \phi(i,r)$ is itself a continuous mesh semimeasure. In that case, the resulting map $\psi = \phi$, and so $M_\phi = M$ is the limiting map from this construction. Therefore, we have an algorithm for left-approximating all the elements of $\mathcal{M}$. Call $(\psi_n)_{n\in\omega}$ the enumeration of all $\mathscr{R}$-computable left-approximators constructed from our enumeration $(\phi_n)_{n\in\omega}$ of all maps ϕ, and $(M_n)_{n\in\omega}$ the corresponding enumeration of all their limiting maps.

We now build another $\mathscr{R}$-computable map $\boldsymbol{\psi}$ as follows:

$$\boldsymbol{\psi} : \omega^2 \to \mathbb{Q} \cap [0,1], \qquad \boldsymbol{\psi}(i,r) := \sum_{n<r} \frac{\psi_n(i,r)}{2^{n+1}},$$

This map $\boldsymbol{\psi}$ is the left-approximator of some lower-semicomputable map $\mathbf{M}$ which satisfies:

$$\mathbf{M} : \omega \to [0,1], \qquad \mathbf{M}(i) := \sum_n \frac{M_n(i)}{2^{n+1}}.$$

We see that $\mathbf{M}$ is a continuous mesh semimeasure on $\mathcal{N}$ because for any $i \in \omega$,

$$\mathbf{M}(i) = \sum_n \frac{M_n(i)}{2^{n+1}} \geq \sum_n \frac{1}{2^{n+1}} \sum_{\mathrm{pred}(i,j)} M_n(j) = \sum_{\mathrm{pred}(i,j)} \sum_n \frac{M_n(j)}{2^{n+1}} = \sum_{\mathrm{pred}(i,j)} \mathbf{M}(j).$$

Moreover,

$$\sum_{\mathrm{root}(i)} \mathbf{M}(i) = \sum_{\mathrm{root}(i)} \sum_n \frac{M_n(i)}{2^{n+1}} = \sum_n \frac{1}{2^{n+1}} \sum_{\mathrm{root}(i)} M_n(i) \leq \sum_n \frac{1}{2^{n+1}} \cdot 1 = 1.$$

Finally, we check that $\mathbf{M}$ is optimal. Fix $M \in \mathcal{M}$, and let $n \in \omega$ be its index in the enumeration of $\mathcal{M}$, giving $M \equiv M_n$. Thus, for any $i \in \omega$,

$$M(i) = M_n(i) \leq 2^{n+1} \cdot \mathbf{M}(i).$$

□

As discussed in Section 1.8, the language of continuous mesh semimeasures may be translated to that of supergales (also see [46]). In particular, given any ρ premeasure on Ω, we might associate to the continuous mesh semimeasure M the *ρ-mesh-supergale*:

$$\delta(i) := \begin{cases} \frac{M(i)}{\rho(\iota(i))} & \rho(\iota(i)) > 0, \\ +\infty & \rho(\iota(i)) = 0. \end{cases}$$

It is straightforward to check that δ satisfies the *ρ-mesh-supergale condition*:

$$\delta(i) \cdot \rho(\iota(i)) \geq \sum_{\text{pred}(i,j)} \delta(j) \cdot \rho(\iota(j)), \quad \text{for each } i \in \omega.$$

In particular, whenever ρ is a computable premeasure, the continuous mesh semimeasure M is lower-semicomputable if and only if its corresponding ρ-mesh-supergale is *constructive*; i.e., lower-semicomputable in the ω-presentation $\mathscr{R}$ of $\mathcal{N}$.

We might also extend the notion of an ρ-mesh-supergale *succeeding* on a point.

Definition 4.3.5. Fix a premeasure ρ on the mesh $\mathcal{N}$. Then a ρ-mesh-supergale δ on $(\Omega, \mathcal{N})$ is said to *succeed on $x \in \Omega$* if there exists an $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ of x such that:

$$\limsup_{n\to\infty} \delta(i_n) = +\infty.$$

This form of success then translates to the following for continuous mesh semimeasures.

Definition 4.3.6. Fix a premeasure ρ on the mesh $\mathcal{N}$. Then a continuous mesh semimeasure M is said to *ρ-succeed on $x \in \Omega$* if there exists an $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ of x with

$$\limsup_{n\to\infty} \left[\frac{M(i_n)}{\rho(\iota(i_n))}\right] = +\infty,$$

where this ratio is defined as $+\infty$ whenever $\rho(\iota(i_n)) = 0$.

These notions of success extend to subsets $X \subseteq \Omega$ by asking that the semimeasure or supergale succeed on *all* elements of X.

Definition 4.3.7. Let M be a lower-semicomputable continuous mesh semimeasure on $\mathcal{N}$. For any index $i \in \omega$, define its *a priori mesh complexity under M* as follows:

$$\mathrm{KM}_M(i) := -\log M(i),$$

or $+\infty$ whenever $M(i) = 0$. Call $\mathrm{KM} \equiv \mathrm{KM}_{\mathbf{M}}$ the *a priori mesh complexity.*

Again, this notion of complexity depends on the ω-presentation $\mathscr{R}$ of $\mathcal{N}$.

Note too that KM has an *invariance* property, stating that any optimal element of $\mathcal{M}$ defines the same *a priori* complexity notion up to an additive constant. Compare this to the invariance of the prefix complexity K up to the choice of universal prefix machine.

4.3.2 Discrete Mesh Semimeasures

Definition 4.3.8. A function $m : \omega \to [0, 1]$ is a *discrete semimeasure* on $(\Omega, \mathcal{N})$ if it satisfies:

$$\sum_{i \in \omega} m(i) \leq 1. \tag{4.4}$$

In words, the *discrete semimeasure condition* in (4.4) simply asks there not be more than unit mass distributed across the countably many mesh elements. We might refer to a discrete semimeasure defined on the ι-indices of the mesh $\mathcal{N}$ to be a *discrete mesh semimeasure* with respect to $\mathcal{N}$.

As in the continuous case, a discrete mesh semimeasure m on $(\Omega, \mathcal{N}, \mathscr{R})$ is *lower-semicomputable* if its lower-graph is computably enumerable relative to the given ω-presentation $\mathscr{R}$. Collect into $\mathfrak{m}$ all the lower-semicomputable discrete mesh semimeasures in $\mathcal{N}$ with respect to the given ω-presentation $\mathscr{R}$. Then, a discrete semimeasure from $\mathfrak{m}$ is *optimal* if it multiplicatively dominates all other semimeasures from $\mathfrak{m}$.

Theorem 4.3.9. *Over a given represented mesh space $(\Omega, \mathcal{N}, \mathscr{R})$, there exists an optimal lower-semicomputable discrete mesh semimeasure.*

Again, we loosely follow the proof structure from Li and Vitányi's Theorem 4.3.1, which shows the existence of an optimal lower-semicomputable discrete semimeasure over some finitely-branching sequence space [29].

Proof. We build our optimal element using left-approximators to the discrete semimeasures in $\mathfrak{m}$. In the first step of the construction, we obtain an enumeration of left-approximators $\widehat{f}$ to the non-negative, lower-semicomputable maps $f \geq 0$. Then, in the next step, each $\widehat{f}$ is turned into a left-approximator for some lower-semicomputable discrete mesh semimeasure. Finally, we take a weighted sum of all these left-approximators to the elements of $\mathfrak{m}$; this defines a computable map which left-approximates an optimal element of $\mathfrak{m}$.

First, note that it is possible to enumerate all $\mathscr{R}$-computable functions of the form $\phi : \omega^2 \to \mathbb{Q} \cap [0, +\infty)$. For any lower-semicomputable map $f : \omega \to [0, \infty)$, the discussion in Section 1.4 guarantees that at least one of the ϕ above will left-approximate f.

Next, use a given ϕ to define another map $\psi : \omega^2 \to \mathbb{R}$ meant to be a left-approximator to some semimeasure in $\mathfrak{m}$. By dovetailing this staged process across the enumeration of all ϕ, we will have described a uniform algorithm for approximating all elements of $\mathfrak{m}$ from below (computably in $\mathscr{R}$), exactly what is needed for building an optimal element of $\mathfrak{m}$.

```
Stage r = 0:
  Initialize ψ(i,0) = 0 for all i ∈ ω;
Stage r > 0:
  If φ(0,r) + ··· + φ(r − 1,r) > 1:
    Set ψ(i,r) = ψ(i,r − 1) for all i ∈ ω;
  Else:
    Set ψ(i,r) := φ(i,r) for all i ∈ ω;
```

In words, for a given stage $r > 0$, we only update ψ when $\phi(\cdot, r)$ assigns no more than unit mass across the mesh elements. Otherwise, we carry over the values of $\psi(\cdot, r-1)$ from the previous stage.

If at any stage the construction attempts to assign more than unit mass across the mesh elements, the resulting ψ will not be updated beyond that stage. Such a map will trivially satisfy $\psi(\cdot, r) : \omega \to \mathbb{R}$ being a discrete mesh semimeasure for each $r \in \omega$.

Otherwise, the map ψ is updated successfully at each stage. We that that the limiting map $m_\phi : i \mapsto \lim_{r\to\infty} \psi(i, r)$ is a discrete mesh semimeasure on $\mathcal{N}$.

Furthermore, for a given $m \in \mathfrak{m}$, there exists an $\mathscr{R}$-computable left-approximator ϕ to m where each map $i \mapsto \phi(i, r)$ is itself a discrete mesh semimeasure. In that case, the resulting map $\psi = \phi$, and so $m_\phi = m$ is the limiting map from this construction. Therefore, we have an algorithm for left-approximating all the elements of $\mathfrak{m}$. Call $(\psi_n)_{n\in\omega}$ the enumeration of all $\mathscr{R}$-computable left-approximators constructed from our enumeration $(\phi_n)_{n\in\omega}$ of all maps ϕ, and $(m_n)_{n\in\omega}$ the corresponding enumeration of all their limiting maps.

We now build another $\mathscr{R}$-computable map $\boldsymbol{\psi}$ as follows:

$$\boldsymbol{\psi} : \omega^2 \to \mathbb{Q} \cap [0,1], \qquad \boldsymbol{\psi}(i,r) := \sum_{n<r} \frac{\psi_n(i,r)}{2^{n+1}},$$

This map $\boldsymbol{\psi}$ is the left-approximator of some lower-semicomputable map $\mathbf{m}$ which satisfies:

$$\mathbf{m} : \omega \to [0,1], \qquad \mathbf{m}(i) := \sum_{n} \frac{m_n(i)}{2^{n+1}}.$$

We see that $\mathbf{m}$ is a discrete mesh semimeasure on $\mathcal{N}$ because for any $i \in \omega$,

$$\sum_i \mathbf{m}(i) = \sum_i \sum_n \frac{m_n(i)}{2^{n+1}} = \sum_n \frac{1}{2^{n+1}} \sum_i m_n(i) \leq \sum_n \frac{1}{2^{n+1}} \cdot 1 \leq 1.$$

Finally, we check that $\mathbf{m}$ is optimal. Fix $m \in \mathfrak{m}$, and let $n \in \omega$ be its index in the enumeration of $\mathfrak{m}$, giving $m \equiv m_n$. Thus, for any $i \in \omega$,

$$m(i) = m_n(i) \leq 2^{n+1} \cdot \mathbf{m}(i).$$

□

4.4 Kolmogorov Complexity on Net Spaces

There are multiple senses in which one might lift the notion of Kolmogorov complexity from finitary objects to *points* in a mesh space. Each is inspired by its corresponding version over either Euclidean or Cantor space. That the various versions agree in the standard Euclidean and sequence spaces follows from the many orderly properties of the prototypical meshes on those spaces. We use this section to argue what properties lend themselves to asymptotic coincidence between these lifts.

4.4.1 Kolmogorov Complexity on Euclidean Space

We continue the discussions of Sections 1.7 and 2.1 to better understand how the lifts of Kolmogorov complexity to Euclidean space might inspire lifts to mesh spaces.

The standard method for lifting prefix complexity to points in a metric space is via a computable, dense subset. Recall that the complexity of an arbitrary subset $X \subseteq \mathbb{R}^m$

was defined as the minimal complexity of a rational in X (or $+\infty$ if no such rational exists). Intuitively, this definition captures the idea that a set is *simple* if it contains a point that is simple to describe, and complex otherwise. Likewise, the complexity of an arbitrary point $x \in \mathbb{R}^m$ is definable to some precision-level $r \in \omega$ as the minimum complexity of a rational in the open 2^{-r}-ball about x. Note that this is equivalent (up to an additive constant) to evaluating the complexity of the dyadic truncation of x up to precision 2^{-r} as in Lemma 1.7.4 and to finding a K-minimizer to $B_{2^{-r}}(x)$ as in Lemma 4.9 of [6].

Note that something similar is true in the conditional case using either the max-min or min-max definitions of conditional prefix complexity $K_{r|s}$ as discussed in Section 2.1. Lemma 1.7.5 shows the ability for dyadic truncation to approximate $K_{r|s}$, and Proposition 2.1.9 does the same via K-minimizers.

Let us now introduce a third manner by which to approximate the lift of prefix complexity to Euclidean space.

In Section 1.2, we defined the collection $\mathcal{Q}^m$ of all dyadic rational cubes over $\mathbb{R}^m$. If $Q \in \mathcal{Q}^m$ is a dyadic cube, let $K(Q)$ denote the prefix complexity $K(i)$ of the index of Q in a standard (computable) enumeration of $\mathcal{Q}^m$. Then, let

$$H_r(x) := \min \left\{ K(Q) : Q \in \mathcal{Q}^m \text{ and } Q \subseteq B_{2^{-r}}(x) \right\},$$

or $+\infty$ if the minimum is taken over the empty set, denote a third lift of complexity to points in $\mathbb{R}^m$. Intuitively, this definition captures the idea that a point is *simple* if there are always sufficiently simple dyadic cubes in its vicinity. Clearly, $H_r(x) \geq^+ K_r(x)$, as the center c_Q of any dyadic cube Q satisfying $Q \subseteq B_{2^{-r}}(x)$ will satisfy $c_Q \in B_{2^{-r}}(x)$. Yet, these notions do not differ significantly.

Proposition 4.4.1. *For all $m \in \omega$, $x \in \mathbb{R}^m$, and $r \in \omega$,*

$$K_r(x) = H_r(x) \pm o(r),$$

where $o(r)$ is a sub-linear term in r independent of x.

Proof. By Lemma 1.7.4, it suffices to show $H_r(x) = K(x \upharpoonright r) \pm o(r)$.

Suppose we are given any dyadic cube $Q \subseteq B_{2^{-r}}(x)$. In particular, its center c_Q is rational and computable from Q, and approximates $x \upharpoonright r$ to a precision $O(2^{-r})$. Thus,

$$K(x \upharpoonright r) \leq K(Q) + O(1) \implies K(x \upharpoonright r) \leq H_r(x) + O(1).$$

In the other direction, notice that $x \upharpoonright r$ is a dyadic rational inside $B_{2^{-r}}(x)$. A simple geometric fact states that there must be one dyadic cube Q of side-length at least $\frac{1}{\sqrt{m}} \cdot 2^{-r}$ which is a subset of the expanded ball: $B_{2 \cdot 2^{-r}}(x \upharpoonright r)$. And by comparing volumes, there can be no more than $(4/\sqrt{m})^m$ many such dyadic cubes. Given $x \upharpoonright r$, we may effectively enumerate all dyadic cubes which are subsets of $B_{2 \cdot 2^{-r}}(x \upharpoonright r)$ and simply specify any one of them by providing its index in that enumeration. Such an index requires no more than $\log((4/\sqrt{m})^m) = O(m \log m)$ many bits to specify. And for any dyadic cube $Q \subseteq B_{2^{-r}}(x)$, it follows from the triangle inequality that $Q \subseteq B_{2 \cdot 2^{-r}}(x \upharpoonright r)$, and so,

$$H_r(x) \leq K(Q) \leq K(x \upharpoonright r) + K(r) + O(m \log m) + O(1).$$

□

Another lift of prefix complexity to Euclidean space may be defined which does not significantly differ from the rest. Let

$$G_r(x) := \min \left\{ K(Q) : x \in Q \in \mathcal{Q}^m \text{ and } \operatorname{diam}(Q) \leq 2^{-r} \right\},$$

or $+\infty$ if the minimum is taken over the empty set. Then, $G_r(x)$ denotes a fourth lift of prefix complexity to points in $\mathbb{R}^m$. Intuitively, this definition captures the idea that a point is *simple* if there are sufficiently small and simple dyadic cubes covering it. Compared to $H_r(x)$, this fourth notion $G_r(x)$ minimizes prefix complexity over a smaller class of dyadic cubes, since $x \in Q$ and $\operatorname{diam}(Q) \leq 2^{-r}$ imply $Q \subseteq B_{2^{-r}}(x)$, but not vice versa. Nevertheless, we observe asymptotic coincidence between the two notions.

Proposition 4.4.2. *For all $m \in \omega$, $x \in \mathbb{R}^m$, and $r \in \omega$,*

$$K_r(x) = G_r(x) \pm o(r),$$

for some fixed $o(r)$ term independent of x and r.

The proof of Proposition 4.4.2 is nearly identical to that of Proposition 4.4.1, noting that exactly one of the at most $(4/\sqrt{m})^m$ dyadic cubes in $B_{2 \cdot 2^{-r}}(x \upharpoonright r)$ must contain x.

4.4.2 Kolmogorov Complexity on Mesh Spaces

In order to lift Kolmogorov complexity to points in a mesh space, we cannot necessarily make use of a computable, dense subset of points or some notion of truncation as is typically done over Euclidean space. Instead, we might measure a point's complexity by the complexity of the mesh elements falling in a ball about the point at some precision-level. This is exactly what $H_r(x)$ and $G_r(x)$ accomplish over Euclidean space with respect to the dyadic cubes net. Over mesh spaces, we will use boldface characters $\mathbf{H}$ and $\mathbf{G}$ for these finite-precision complexity notions.

Definition 4.4.3. For any $x \in \Omega$ and $r \in \omega$, define the *vicinity complexity of x up to precision-level r with respect to the mesh $\mathcal{N}$* as:

$$\mathbf{H}_r(x) := \min \{K(i) : \iota(i) \subseteq B_{2^{-r}}(x)\}.$$

Definition 4.4.4. For any $x \in \Omega$ and $r \in \omega$, define the *covering complexity of x up to precision-level r with respect to the mesh $\mathcal{N}$* as:

$$\mathbf{G}_r(x) := \min \left\{K(i) : x \in \iota(i) \wedge \operatorname{diam}(i) \leq 2^{-r}\right\}.$$

In order to show asymptotic coincidence between these two notions of complexity, we will need to ask more from the mesh.

Definition 4.4.5. If $\mathcal{N}$ is a net on Ω, we call $\mathcal{N}$ a *grate* on Ω if it further satisfies: there exist both a function $f : \omega \to [1, +\infty)$ and an $o(r)$ function $g : \omega \to \mathbb{R}$ such that for all precision-levels $r \in \omega$:

(**G5**) For all $x \in \Omega$, there exists $N \in \mathcal{N}$ such that $x \in N$ and $\operatorname{diam}_d(N) \in \left[2^{-r-f(r)}, 2^{-r}\right]$;

(**G6**) For all $N \in \mathcal{N}$ satisfying $\operatorname{diam}_d(N) \leq 2^{-r}$, it holds that

$$\log \left| \left\{ N' \in \mathcal{N} : N \cap N' \neq \emptyset \text{ and } \operatorname{diam}_d(N') \in \left[2^{-r-f(r)}, 2^{-r}\right] \right\} \right| \leq g(r).$$

In words, the grate axioms ask that:

(**G5**) For each point there is a grate element covering it at every precision-level;

(**G6**) For each grate element, there are no more than sub-exponentially many other grate elements non-trivially intersecting it at each smaller precision-level.

When $\mathcal{N}$ is a grate, we should additionally include f and g in the ω-presentation $\mathscr{R}$ of the grate, as well as another relation encoding intersection between grate elements:

- $\mathrm{int} : \omega^2 \to \{\top, \bot\}$ is a relation on pairs of indices encoding the non-trivial intersection relation; i.e., if $N_i, N_j \in \mathcal{N}$ satisfy $\iota(i) = N_i$ and $\iota(j) = N_j$, then,

$$\mathrm{int}(i, j) \iff N_i \cap N_j \neq \emptyset.$$

For any grate $\mathcal{N}$ on a metric space (Ω, d) represented by the ω-presentation $\mathscr{R}$ as described above, we may call the pair $(\Omega, \mathcal{N})$ a *grate space* and the tuple $(\Omega, \mathcal{N}, \mathscr{R})$ a *represented grate space*.

We now show that vicinity and covering complexities agree asymptotically on grate spaces. Our proof of this agreement relies on the density of the mesh along each precision-level.

Proposition 4.4.6. *For any grate space $(\Omega, \mathcal{N})$, $x \in \Omega$, and $r \in \omega$,*

$$\mathbf{H}_r(x) = \mathbf{G}_r(x) \pm o(r),$$

for some sublinear term $o(r)$ independent of x.

Proof. Let ι index $\mathcal{N}$ while respecting containment. First, we immediately have that $\mathbf{H}_r(x) \leq \mathbf{G}_r(x)$, since $x \in \iota(i)$ and $\mathrm{diam}(i) \leq 2^{-r}$ together imply $\iota(i) \subseteq B_{2^{-r}}(x)$ via the triangle inequality.

In the other direction, let $i \in \omega$ satisfy the vicinity condition $\iota(i) \subseteq B_{2^{-r}}(x)$. Recall that for grate spaces, we have augmented their ω-presentation to encode $f(r)$ and the non-trivial intersection relation, int. So, using i and r, we may computably enumerate in the ω-presentation of the grate $\mathcal{N}$ all indices $j \in \omega$ satisfying:

$$\mathrm{int}(i, j) \quad \text{and} \quad \mathrm{diam}(j) \in \left[2^{-r-f(r)}, 2^{-r}\right].$$

By (**G5**), one such $j = j^*$ further satisfies $x \in \iota(j^*)$. And by construction, $\mathrm{diam}(j^*) \leq 2^{-r}$, qualifying j^* for the covering condition that defines $\mathbf{G}_r(x)$. Moreover, (**G6**) guarantees that this enumeration is not too large: there are no more than $2^{g(r)}$ many indices in the enumeration, where $g(r)$ is the $o(r)$ function guaranteed in the grate axioms. Therefore, using a two-part description for j^* with respect to this enumeration, we conclude:

$$K(j^*) \leq K(i) + \lceil g(r) \rceil + K(r) + O(1) = K(i) + o(r),$$

so, $\mathbf{G}_r(x) \leq \mathbf{H}_r(x) + o(r)$, as desired. □

We might take either one of these quantities $\mathbf{G}_r$ or $\mathbf{H}_r$ as our primary definition of prefix complexity over a mesh space. Anticipating the results of the following section, we select $K_r := \mathbf{G}_r$ as the *prefix complexity of x with respect to the mesh $\mathcal{N}$.*

Example 4.4.7. Continuing with the Lebesgue measure and the dyadic cube net $\mathcal{Q}^m$ on Euclidean space as introduced in Example 4.1.8, we show that $(\mathbb{R}^m, \mathcal{Q}^m)$ is a grate space. Note that the non-trivial intersection relation is also easily computable from the enumeration of $\mathcal{Q}^m$. Fixing $r \in \omega$ and $x \in \mathbb{R}^m$, it is clear that there exists a dyadic cube Q of side-length $2^{-r-\lceil \log \sqrt{m} \rceil}$ satisfying $x \in Q$. The diameter of this cube is:

$$\operatorname{diam}_d(Q) = 2^{-r-\lceil \log \sqrt{m} \rceil} \cdot \sqrt{m} \leq 2^{-r}.$$

Setting $f(r) := \lceil \log \sqrt{m} \rceil$, we conclude (**G5**). Note that non-trivial intersection between dyadic cubes implies comparability: one is a subset of the other. So, we may also bound from above the number of dyadic cubes having diameter in the interval $[2^{-r-f(r)}, 2^{-r}]$ and non-trivially intersecting Q:

$$\sum_{i=1}^{\lceil f(r) \rceil} (2^m)^i \leq 2^{m(\lceil f(r) \rceil + 1)} = 2^{\left(\lceil \log \sqrt{m} \rceil + 1\right)}.$$

Setting $g(r) := \lceil \log \sqrt{m} \rceil + 1 = O(1)$, we conclude (**G6**) as well. Thus, $(\mathbb{R}^m, \mathcal{Q}^m)$ is a grate space; so, by Proposition 4.4.6, $\mathbf{H}_r(x) = \mathbf{G}_r(x) \pm o(r)$ for all $x \in \mathbb{R}^m$, a fact we have already seen as a consequence of Propositions 4.4.1 and 4.4.2. We further have that the non-trivial intersection relation is equivalent to the containment relation for this grate, and $f(r)$ is a computable function, so $\mathcal{Q}^m$ has a computable ω-presentation, meaning $\mathcal{Q}^m$ qualifies as an *effective grate* on $\mathbb{R}^m$. So $\mathbb{R}^m$ with $\mathcal{Q}^m$ is in fact a *computable grate space.*

4.4.3 Optimality of Outer Measures on Computable Net Spaces

We still have not explicitly defined the Kolmogorov complexity of an arbitrary *subset* of a mesh space. For instance, it is possible to view vicinity complexity as induced by the following complexity notion on subsets: for any $X \subseteq \Omega$,

$$\mathbf{H}(X) := \min\{K(i) : \iota(i) \subseteq X\}, \quad \text{so} \quad \mathbf{H}_r(x) := \mathbf{H}(B_{2^{-r}}(x)).$$

However, this complexity notion is not particularly useful. For instance, N. Lutz argues how the map $\kappa(X) := 2^{-K(X)}$, which possesses many algorithmic optimality properties, is an outer measure on $\mathbb{R}^m$ [35, 36]. In contrast, $X \mapsto 2^{-\mathbf{H}(X)}$ is not even necessarily countably subadditive. This is because little is guaranteed about how the mesh behaves topologically: while each subset $X_i \subseteq \Omega$ in a countable sequence could contain only mesh elements of very high complexity, their union $\bigcup_i X_i$ could contain a much larger mesh element of low complexity. However, on computable net spaces, it is indeed possible to define a reasonably well-behaved complexity notion for arbitrary subsets.

Let us now extend some of the algorithmic optimality results found in [35] to any computable net space $(\Omega, d, \mathcal{N}, \mathscr{R}, \alpha)$. We use the notation $\mathcal{P}^{<\omega}(\alpha)$ to refer to the collection of finite subsets of the computable dense sequence α. And, given an element $N \in \mathcal{N}$, denote by $K_{\mathcal{N}}(N) := K(i)$ the *net-complexity* of N, evaluated on its index: $\iota(i) = N$. The following three properties of an outer measure will comprise our notion of *global optimality*, an optimality notion similar to that of the optimal lower-semicomputable discrete semimeasure $\mathbf{m}$.

Definition 4.4.8. An outer measure μ on Ω is *finitely supported* on α if, for every $\varepsilon > 0$, there exists $A \in \mathcal{P}^{<\omega}(\alpha)$ such that $\mu(\Omega \setminus A) < \varepsilon$.

Definition 4.4.9. An outer measure μ on Ω is *strongly finite* if μ is supported on α and

$$\sum_{q \in \alpha} \mu(\{q\}) < +\infty.$$

Definition 4.4.10. An outer measure μ on Ω is *lower-semicomputable* if it is finitely supported on α and there is a computable function (witness) $\widehat{\mu} : \mathcal{P}^{<\omega}(\alpha) \times \omega \to \mathbb{Q} \cap [0, +\infty)$ such that for all $A \in \mathcal{P}^{<\omega}(\alpha)$ and $r \in \omega$,

$$\widehat{\mu}(A, r) \leq \widehat{\mu}(A, r+1) \leq \mu(A), \quad \text{and} \quad \lim_{r \to \infty} \widehat{\mu}(A, r) = \mu(A).$$

Let Θ be the set of all strongly finite, lower-semicomputable outer measures on Ω.

Definition 4.4.11. An outer measure μ on Ω is *globally optimal* if $\mu \in \Theta$ and multiplicatively dominates any other $\theta \in \Theta$, i.e., for any $\theta \in \Theta$, there exists $\beta \in (0, +\infty)$ such that for each $X \subseteq \Omega$,

$$\mu(X) \geq \beta \cdot \theta(X).$$

By the same arguments found in [35] (see Lemma 3.3 and Theorem 3.4), there exist globally optimal outer measures on computable metric spaces.

Proposition 4.4.12. *Over any computable metric space (Ω, d, α), there exists a globally optimal strongly finite, lower-semicomputable outer measure $\boldsymbol{\theta}$.*

Their construction follows a similar format as Theorems 4.3.4 and 4.3.9, producing an optimal outer measure as a weighted sum of all those found in Θ. We next define the *prefix complexity of an arbitrary subset* $X \subseteq \Omega$ in a computable metric space (Ω, d, α) as:

$$\mathbf{K}(X) := \min \left\{ \mathbf{K}(q) : q \in X \cap \alpha \right\}, \tag{4.5}$$

where $\mathbf{K}(q)$ is defined as $K(i)$, and $q = \alpha_i = \chi(i)$ has index $i = \text{index}(q)$ in the enumeration of the computable dense subset.

From (4.5), we define what we will refer to as the *prefix measure* of a subset $X \subseteq \Omega$ as follows:

$$\boldsymbol{\kappa}(X) := 2^{-\mathbf{K}(X)}.$$

It is easy to check that $\boldsymbol{\kappa}$ is indeed an outer measure on Ω, as well as lower-semicomputable and strongly finite. That $\boldsymbol{\kappa}$ is lower-semicomputable follows from the upper semi-computability of K. That $\boldsymbol{\kappa}$ is finitely supported on α follows from the fact that $A_\varepsilon := \left\{ q \in \alpha : \mathbf{K}(q) \leq \log_2 \frac{1}{\varepsilon} \right\}$ is a finite subset of α accounting for all but at most ε amount of the measure that $\boldsymbol{\kappa}$ places on Ω. And that $\boldsymbol{\kappa}$ is strongly finite then follows from the Kraft Inequality 1.6.4:

$$\sum_{q \in \alpha} \boldsymbol{\kappa}(\{q\}) = \sum_{q \in \alpha} 2^{-\mathbf{K}(\{q\})} \leq 1.$$

It turns out that $\boldsymbol{\kappa}$ is *not* globally optimal over Ω. To see this, let us first define the subset-analogues of each of the equivalent quantities in Levin's Coding Theorem 1.8.2:

$$\begin{aligned}
\boldsymbol{\mu}(X) &:= \sum_{q \in X \cap \alpha} \boldsymbol{\mu}(q), \\
\mathbf{Q}(X) &:= \sum_{q \in X \cap \alpha} \mathbf{Q}(q) = \sum_{\mathbf{U}_{\mathbf{PF}}(\pi)\downarrow \in X \cap \alpha} 2^{-\operatorname{len}(\pi)}, \\
\mathbf{R}(X) &:= \sum_{q \in X \cap \alpha} \mathbf{R}(q) = \sum_{q \in X \cap \alpha} 2^{-\mathbf{K}(q)},
\end{aligned}$$

where $\boldsymbol{\mu}(q)$, $\mathbf{Q}(q)$, and $\mathbf{R}(q)$ are evaluated as $\mathbf{m}(i)$, $Q(i)$, and $R(i)$, respectively, for index i of q in α. Notice that for any $X \subseteq \Omega$,

$$\boldsymbol{\kappa}(X) \leq \mathbf{R}(X) \leq \mathbf{Q}(X).$$

The same discussion in [35] works to show that each of these maps defined above is a strongly finite, lower-semicomputable outer measure on Ω. In fact, each is globally optimal.

Proposition 4.4.13. *Each of* $\boldsymbol{\mu}$, $\mathbf{Q}$, *and* $\mathbf{R}$ *is globally optimal on* Ω.

Proof. By Levin's Coding Theorem 1.8.2, each of $\boldsymbol{\mu}$, $\mathbf{Q}$, and $\mathbf{R}$ multiplicatively dominate the others as functions on α. So, by definition, $\boldsymbol{\mu}$, $\mathbf{Q}$, and $\mathbf{R}$ dominate each other as outer measures from Θ. Moreover, for any $\theta \in \Theta$, we claim that $\boldsymbol{\mu}$ dominates θ. Note that θ induces the discrete semimeasure: $i \mapsto \theta(\{\alpha_i\})$. So, by the assumptions on θ as well as the optimality of $\mathbf{m}$ as a discrete semimeasure, we conclude there exists $\beta \in (0, +\infty)$ such that for any $X \subseteq \Omega$:

$$\begin{aligned}
\theta(X) &\leq \theta(X \cap \alpha) + \theta(X \setminus \alpha) \\
&\leq \sum_{q \in X \cap \alpha} \theta(\{q\}) + \theta(\Omega \setminus \alpha) \\
&\leq \frac{1}{\beta} \sum_{q \in X \cap \alpha} \boldsymbol{\mu}(\{q\}) + 0 \\
&= \frac{1}{\beta} \cdot \boldsymbol{\mu}(X).
\end{aligned}$$

□

Lemma 4.4 of [35] may be adapted to (Ω, d, α) to find a family of subsets of Ω exhibiting $\boldsymbol{\kappa}$ failing to dominate $\boldsymbol{R}$. Given a constant $\beta > 0$, they construct a set $X_\beta = \{q \in \alpha : K(q) > \eta\}$ for some sufficiently large constant $\eta > 0$ satisfying $\eta - O(\log \eta) < \log \beta$ and exhibiting $\boldsymbol{\kappa}(X_\beta) < \beta \cdot \mathbf{R}(X_\beta)$. Notice that by definition, $\boldsymbol{\kappa}(X_\beta) < 2^{-\eta}$, while

$$\begin{aligned}
\mathbf{R}(X_\beta) = \sum_{q \in X_\beta \cap \alpha} 2^{-K(q)} &\geq \sum_{\substack{q \in X_\beta \cap \alpha \\ K(q) < \eta + O(\log \eta)}} 2^{-K(q)} \\
&\geq 2^{-(\eta + O(\log \eta))} \cdot \left| \{q \in \alpha : K(q) \in (\eta, \eta + O(\log \eta))\} \right|.
\end{aligned}$$

Then, by a short counting argument, they lower-bound the cardinality of the set appearing above by 2^η, giving $\mathbf{R}(X_\beta) \geq 2^{-O(\log \eta)}$. Thus, $\boldsymbol{\kappa}(X_\beta) < \beta \cdot \mathbf{R}(X_\beta)$, showing $\boldsymbol{\kappa}$ is not globally optimal.

Definition 4.4.14. Let μ and θ be outer measures on Ω and $\mathcal{X}$ be a layered disjoint system over Ω (in the sense of [35]). Then μ is said to *dominate* θ *on* $\mathcal{X}$ if there exists a map $\beta : \omega \to (0, +\infty)$ such that $-\log \beta(r) = o(r)$ as $r \to \infty$, and for each $r \in \omega$ and

$X \in \mathscr{X}^{(r)}$ in the r-th layer of $\mathscr{X}$,

$$\mu(X) \geq \beta(r) \cdot \theta(X).$$

Recall that for any layered-disjoint net $\mathcal{N}$, the sequence of rank layers $(\mathcal{N}^{(r)})_{r\in\omega}$ induces a layered disjoint system in the sense of [35]. So we may discuss domination on any layered-disjoint net.

Definition 4.4.15. An outer measure μ on net space $(\Omega, \mathcal{N})$ is *locally optimal* if $\mu \in \Theta$ and for each $\theta \in \Theta$, μ dominates θ on $\mathcal{N}$.

Theorem 4.4.16. *Over any computable net space* $(\Omega, d, \mathcal{N}, \mathscr{R}, \alpha)$ *with a layered-disjoint net* $\mathcal{N}$*, we have that* $\boldsymbol{\kappa}$ *is locally optimal.*

Proof. From before, $\boldsymbol{\kappa} \in \Theta$. We claim for each $\theta \in \Theta$, $\boldsymbol{\kappa}$ dominates θ on $\mathcal{N}$. Denote for each $i \in \omega$:

$$m_\theta(i) := \theta(\{\alpha_i\}).$$

Then m_θ is a lower-semicomputable discrete net semimeasure and hence multiplicatively dominated by $\mathbf{m}$: let $\gamma \in (0, +\infty)$ satisfy $\mathbf{m}(i) \geq \gamma \cdot m_\theta(i)$ for each $i \in \omega$. Recall the notation $\mathcal{N}^{(r)}$ for the collection of all net elements with rank r. Then $(\mathcal{N}^{(r)} : r \in \omega)$ is a layered disjoint system, and hence obeys the *LDS Coding Theorem* (Theorem 3.1 of [6]). In particular, we have for each $N \in \mathcal{N}^{(r)}$:

$$K_{\mathcal{N}}(N) \leq -\log \mathbf{Q}(N) + K(r) + O(1) \leq -\log \boldsymbol{\mu}(N) + K(r) + O(1). \tag{4.6}$$

Call $\beta(r) = \gamma \cdot 2^{-K(r)-O(1)}$ using the same additive constant as in the LDS Coding Theorem. Notice that indeed $-\log \beta(r) = o(r)$. Then for each $N \in \mathcal{N}^{(r)}$, we use that θ is both countably subadditive and finitely supported on α, as well as (4.6), to confirm:

$$\begin{aligned}
\boldsymbol{\kappa}(N) &= 2^{-\mathbf{K}(N)} \\
&\geq 2^{-K_{\mathcal{N}}(N)} \\
&\geq 2^{-K(r)-O(1)} \cdot \boldsymbol{\mu}(N) \\
&= 2^{-K(r)-O(1)} \cdot \sum_{q \in N \cap \alpha} \boldsymbol{\mu}(q) \\
&\geq 2^{-K(r)-O(1)} \cdot \sum_{q \in N \cap \alpha} \gamma \cdot m_\theta(q)
\end{aligned}$$

$$= \beta(r) \cdot \sum_{q \in N \cap \alpha} \theta(\{q\})$$
$$\geq \beta(r) \cdot \theta(N).$$

□

We now ask how the complexity $K_{\mathcal{N}}(N)$ of the address of an element N with respect to the tree structure of $\mathcal{N}$ compares to the complexity $\mathbf{K}(N)$ as a subset of Ω? We have already used the fact that $\mathbf{K}(N) \leq K_{\mathcal{N}}(N)$: if $\iota(i) = N$, then $\alpha_i \in N$, and $\mathbf{K}(\alpha_i) = K(i) = K_{\mathcal{N}}(N)$. The other direction follows given some stronger assumptions about the net.

Proposition 4.4.17. *Let $(\Omega, d, \mathcal{N}, \mathscr{R}, \alpha)$ be a computable grate space. Then, for each $N \in \mathcal{N}$,*

$$K_{\mathcal{N}}(N) = \mathbf{K}(N) \pm o\left(-\log \operatorname{diam}_d(N)\right),$$

where the $o(\cdot)$ term vanishes like $\frac{o(-\log \operatorname{diam}_d(N))}{\log \operatorname{diam}_d(N)} \to 0$ as $\operatorname{diam}_d(N) \to 0$.

Proof. Let f and g be as in the definition of a grate space. Take $N \in \mathcal{N}$ and $r \in \omega$ such that $\operatorname{diam}_d(N) \in [2^{-r-f(r)}, 2^{-r}]$. Further suppose $i, j \in \omega$ are such that $\iota(i) = N$ and $\alpha_j \in N$ with minimal complexity $K(j)$. We may compute i from j as follows: by (**G6**), there are no more than $2^{g(r)}$ net elements non-trivially intersecting $\iota(j)$ with diameter in this same range $[2^{-r-f(r)}, 2^{-r}]$. Thus, N may be computed by specifying an index in this collection, giving

$$K_{\mathcal{N}}(N) = K(i) \leq K(j) + \lceil g(r) \rceil + 1 + O(1) \leq \mathbf{K}(N) + o(-\log \operatorname{diam}_d(N)).$$

□

Finally, we ask whether $\mathbf{K}$ is compatible with $\mathbf{G}$ and $\mathbf{H}$ as complexity notions on points. In particular, we extend $\mathbf{K}$ to arbitrary points of the metric space at fixed precision-levels:

$$\mathbf{K}_r(x) := \mathbf{K}(B_{2^{-r}}(x)).$$

Note that this complexity notion plays an analogous role to $C_\delta(x)$ in [33], where $\delta = 2^{-r}$.

Proposition 4.4.18. *For any grate space* $(\Omega, \mathcal{N})$*,* $x \in \Omega$*, and* $r \in \omega$*,*

$$\mathbf{K}_r(x) = \mathbf{H}_r(x) \pm o(r) = \mathbf{G}_r(x) \pm o(r),$$

for some sublinear terms $o(r)$ *independent of* x*.*

The second equality above restates Proposition 4.4.6. And the proof of the above result mimics those of Propositions 4.4.1 and 4.4.17.

4.5 Effective Dimension on Net Spaces

In this section we extend the results of Section 1.9 by introducing and comparing multiple real-valued functions on a mesh space which attempt to capture the information density of individual points. For the rest of the section, fix a represented mesh space $(\Omega, \mathcal{N}, \mathscr{R})$. For notational convenience, if dim is a dimension on subsets of Ω, we will sometimes use $\dim(x)$ for $\dim(\{x\})$.

4.5.1 Local Dimension of Semimeasures

From any mesh semimeasure, we may produce a Billingsley-type, algorithmic, pointwise dimension. Define the *local dimension* of a (discrete or continuous) mesh semimeasure as follows.

Definition 4.5.1. If m is a (discrete or continuous) mesh semimeasure on $\mathcal{N}$, the *(lower) local dimension of* $x \in \Omega$ *under* m *with respect to* $\mathcal{N}$ is:

$$\dim_{\mathrm{loc}} \mathsf{m}(x) = \inf \left\{ \liminf_{n \to \infty} \frac{\log \mathsf{m}(i_n)}{\log \operatorname{diam}(i_n)} : (i_n)_n \in \mathcal{R}(x) \right\},$$

where the ratio is defined to be zero whenever $\operatorname{diam}(i_n) = 0$. And if $X \subseteq \Omega$, then the *(lower) local dimension of* m *on* X *with respect to* $\mathcal{N}$ is:

$$\dim_{\mathrm{loc}} \mathsf{m}(X) = \sup_{x \in X} \dim_{\mathrm{loc}} \mathsf{m}(x).$$

Given a represented mesh space $(\Omega, \mathcal{N}, \mathscr{R})$, we let $\mathbf{M}_{\mathscr{R}}$ and $\mathbf{m}_{\mathscr{R}}$ be the optimal semimeasures constructed in Theorems 4.3.4 and 4.3.9, respectively. As $\mathscr{R}$ is fixed in this section, we may drop the reference to it in the notation, writing $\mathbf{M}$ and $\mathbf{m}$.

Just as $\mathbf{M}$ is a multiplicatively-largest, lower-semicomputable (in $\mathscr{R}$) continuous mesh semimeasure on $\mathcal{N}$, its local dimension $\dim_{\text{loc}} \mathbf{M}$ is the multiplicatively-largest local dimension arising from lower-semicomputable continuous mesh semimeasures. We call $\dim_{\text{loc}} \mathbf{M}(\cdot)$ the *(lower) continuous local dimension (with respect to mesh* $(\Omega, \mathcal{N}, \mathscr{R})$*)*, which is invariant under any choice of optimal semimeasure $\mathbf{M}$.

Similarly, as $\mathbf{m}$ is the multiplicatively-largest, lower-semicomputable (in $\mathscr{R}$) discrete mesh semimeasure on $\mathcal{N}$, its local dimension $\dim_{\text{loc}} \mathbf{m}$ is the multiplicatively-largest local dimension arising from lower-semicomputable discrete mesh semimeasures. We call $\dim_{\text{loc}} \mathbf{m}(\cdot)$ the *(lower) discrete local dimension (with respect to mesh* $(\Omega, \mathcal{N}, \mathscr{R})$*)*, which is invariant under any choice of optimal semimeasure $\mathbf{m}$.

4.5.2 Constructive Dimension

Following J. Lutz's work over Cantor space in [31], we define the *constructive dimension* $\text{cdim}(x)$ of a point x as the infimum over all $s \geq 0$ for which there exists a constructive s-mesh-supergale succeeding on x.

Definition 4.5.2. For any subset $X \subseteq \Omega$, the *(weak) constructive dimension of* X *(with respect to* $(\Omega, \mathcal{N}, \mathscr{R})$*)* is defined as:

$$\begin{aligned} \text{cdim}(X) &= \inf \{s \geq 0 : \text{there is an } \mathscr{R}\text{-constructive } s\text{-mesh-supergale succeeding on } X\} \\ &= \inf \left\{ s \geq 0 : \begin{array}{c} \text{there is am } \mathscr{R}\text{-lower-semicomputable continuous} \\ \text{mesh semimeasure } s\text{-succeeding on } X \end{array} \right\}. \end{aligned}$$

Note that we may as well define constructive dimension with respect to any optimal lower-semicomputable continuous mesh semimeasure, since if some $M \in \mathcal{M}$ were to indeed s-succeed on X, then so too would $\mathbf{M}$ s-succeed on X:

$$\begin{aligned} \text{cdim}(X) &= \inf \left\{ s \geq 0 : (\exists M \in \mathcal{M})(\forall x \in X)(\exists (i_n)_n \in \mathcal{R}(x)) \left[\limsup_{n\to\infty} \frac{M(i_n)}{\text{diam}(i_n)^s} = +\infty \right] \right\} \\ &= \inf \left\{ s \geq 0 : (\forall x \in X)(\exists (i_n)_n \in \mathcal{R}(x)) \left[\limsup_{n\to\infty} \frac{\mathbf{M}(i_n)}{\text{diam}(i_n)^s} = +\infty \right] \right\}. \end{aligned}$$

As the success notions for continuous semimeasures and supergales coincide, we see:

Proposition 4.5.3. *For any represented mesh space* $(\Omega, \mathcal{N}, \mathscr{R})$ *and any subset* $X \subseteq \Omega$,

$$\dim_{\text{loc}} \mathbf{M}(X) = \text{cdim}(X).$$

4.5.3 Incompressibility Dimension

Definition 4.5.4. The *(lower) incompressibility ratio of a subset $X \subseteq \Omega$ (with respect to* $(\Omega, \mathcal{N}, \mathscr{R})$*)* is defined in two manners:

$$\underline{\gamma}(X) := \sup_{x\in X} \liminf_{r\to\infty} \frac{\mathbf{G}_r(x)}{r}, \quad \text{and} \quad \underline{\eta}(X) := \sup_{x\in X} \liminf_{r\to\infty} \frac{\mathbf{H}_r(x)}{r}.$$

Over net spaces, the incompressibility ratio $\underline{\gamma}$ and the local continuous dimension $\dim_{\text{loc}} \mathbf{M}$ coincide.

Theorem 4.5.5. *For any represented net space $(\Omega, \mathcal{N}, \mathscr{R})$ and subset $X \subseteq \Omega$,*

$$\dim_{\text{loc}} \mathbf{M}(X) = \underline{\gamma}(X).$$

Proof. Let us show that $\dim_{\text{loc}} \mathbf{M}(X) \geq \underline{\gamma}(X)$. Take any rational $s > \dim_{\text{loc}} \mathbf{M}(X)$. For each $\ell \in \omega$, define the set of indices:

$$A_\ell := \left\{ i \in \omega : \frac{\mathbf{M}(i)}{\text{diam}(i)^s} \geq 2^\ell \right\}.$$

Then, for any $\ell \in \omega$, since $\mathbf{M}$ is $[0,1]$-valued, it holds that for any $i \in A_\ell$,

$$\text{diam}(i) \leq \left(\mathbf{M}(i) \cdot 2^{-\ell}\right)^{1/s} \leq 2^{-\ell/s}. \tag{4.7}$$

Claim: If $\mathcal{E} \subseteq \omega$ is a prefix-free collection of indices of net elements (i.e., having a prefix-free collection of addresses in the forest $\mathcal{G}$ associated to $\mathcal{N}$), then for any $r \in \omega$,

$$|B_r \cap \mathcal{E}| \leq 2^{-\ell+rs+s}, \quad \text{where} \quad B_r := \left\{ i \in A_\ell : \text{diam}(i) > 2^{-r-1} \right\}.$$

Proof. For each $k \in \omega$ satisfying root(k), let $\mathcal{E}^k$ denote the collection of indices in $\mathcal{E}$ for subsets of the root element with index k, i.e.,

$$\mathcal{E}^k := \{ i \in \mathcal{E} : \text{in}(i,k) \}.$$

Then, by $\mathcal{E}^k \sqsubseteq_{\text{PF}} k$, the Generalized Kraft Inequality (Proposition 4.3.2), the continuous net semimeasure property (4.3), and the definition of B_r:

$$\begin{aligned} 1 \geq \sum_{\text{root}(k)} \mathbf{M}(k) &\geq \sum_{\text{root}(k)} \sum_{i \in B_r \cap \mathcal{E}^k} \mathbf{M}(i) \\ &\geq \sum_{\text{root}(k)} \sum_{i \in B_r \cap \mathcal{E}^k} 2^{\ell} \cdot \operatorname{diam}(i)^s \\ &\geq \sum_{\text{root}(k)} \sum_{i \in B_r \cap \mathcal{E}^k} 2^{\ell} \cdot 2^{s(-r-1)} = |B_r \cap \mathcal{E}| \cdot 2^{\ell - rs - s}. \end{aligned}$$

□

One may enumerate the elements of A_ℓ effectively in $\mathscr{R}$. Build a set $\mathcal{E}_\ell \subseteq A_\ell$ of indices of mutually-incomparable net elements by only enumerating into $\mathcal{E}_\ell$ those newly enumerated indices of A_ℓ incomparable to all indices already enumerated into $\mathcal{E}_\ell$. For any $i \in \mathcal{E}_\ell$ with $\operatorname{diam}(i) > 0$, (4.7) implies that there exists some natural number $r \geq \lceil \ell/s \rceil$ with $2^{-r-1} < \operatorname{diam}(i) \leq 2^{-r}$, so $i \in B_r$. Using the two-part description of $i \in B_r \cap \mathcal{E}_\ell$, we see by the claim above that:

$$K^{\mathscr{R}}(i) \leq rs + s - \ell + O(\log r) + O(\log \ell).$$

Fix $x \in X$. By assumption, $\mathbf{M}$ s-succeeds on x, so x has an $\mathcal{N}$-name intersecting A_ℓ for arbitrary large $\ell \in \omega$. If that name settles on a net element of diameter zero, then $\mathbf{G}_r(x) = O(1)$ for sufficiently large r. Otherwise, for every $\ell \in \omega$, there exists $i_\ell \in \mathcal{E}_\ell$ with $\operatorname{diam}(i_\ell) > 0$ such that $x \in \iota(i_\ell)$, giving,

$$\liminf_{r \to \infty} \frac{\mathbf{G}_r(x)}{r} \leq \liminf_{r \to \infty} \frac{rs + s - \ell + O(\log r) + O(\log \ell)}{r} \leq s.$$

Taking a supremum over all $x \in X$, we finish the proof in this direction.

In the other direction, take rationals $s > s' > s'' > \underline{\gamma}(X)$. Put

$$A := \left\{ i \in \omega : K^{\mathscr{R}}(i) \leq -s' \cdot \log \operatorname{diam}(i) \right\},$$

where $\log(0)$ is defined to be $-\infty$. This set is c.e. in $\mathscr{R}$. Define the map,

$$M(i) := \sum_{\substack{j \in A \\ \text{in}(j,i)}} \operatorname{diam}(j)^{s'}.$$

In words, the sum defining M considers all net elements which are subsets of the i-th net element and which have bounded complexity. This map is well-defined (and, in fact, satisfies the bounding property (4.3)) by the Kraft Inequality 1.6.4:

$$\sum_{\mathrm{root}(k)} M(k) \leq \sum_{j \in A} \mathrm{diam}(j)^{s'} \leq \sum_{j} 2^{-K^{\mathscr{R}}(j)} \leq 1.$$

We claim M is an $\mathscr{R}$-lower-semicomputable continuous net semimeasure.

Lower-semicomputability is clear from having computable access to the containment relation and diam, as well as by $K^{\mathscr{R}}(i)$ being upper-semicomputable in $\mathscr{R}$. The continuous net semimeasure property (4.2) holds since M satisfies the recursive rule:

$$\begin{aligned} M(i) &= \sum_{\mathrm{pred}(i,j)} M(j) + \begin{cases} \mathrm{diam}(i)^{s'} & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases} \\ &\geq \sum_{\mathrm{pred}(i,j)} M(j). \end{aligned}$$

This follows from the net axiom (**N4**). So, $M \in \mathcal{M}$. And for any $i \in A$,

$$M(i) \geq \mathrm{diam}(i)^{s'}. \tag{4.8}$$

Fix $x \in X$. By assumption, there are infinitely many $r \in \omega$ for which $\mathbf{G}_r(x) \leq rs''$. For each such r, fix $i_r \in \omega$ such that $x \in \iota(i_r)$, $\mathrm{diam}(i_r) \leq 2^{-r}$ and $K^{\mathscr{R}}(i_r) \leq rs''$. If any $\mathrm{diam}(i_r) = 0$, we are done. Otherwise, these conditions imply $i_r \in A$, so by (4.8),

$$\frac{M(i_r)}{\mathrm{diam}(i_r)^s} \geq \frac{\mathrm{diam}(i_r)^{s'}}{\mathrm{diam}(i_r)^s} = \mathrm{diam}(i_r)^{s'-s} \geq 2^{r(s-s')} \to \infty \text{ as } r \to \infty.$$

So M s-succeeds on x, and so too does $\mathbf{M}$. Therefore, $\mathbf{M}$ s-succeeds on all of X. □

Corollary 4.5.6. *For any represented net space $(\Omega, \mathcal{N}, \mathscr{R})$ and subset $X \subseteq \Omega$,*

$$\dim_{\mathrm{loc}} \mathbf{m}(X) = \dim_{\mathrm{loc}} \mathbf{M}(X).$$

Therefore, the universal semimeasures in both the discrete and continuous senses produce the same local dimension notion over Ω.

Proof. By Theorem 4.5.5, it suffices to show that $\dim_{\mathrm{loc}} \mathbf{m}(X) = \underline{\gamma}(X)$. This latter equality holds even over mesh spaces.

Suppose $\underline{\gamma}(X) < s \in \mathbb{Q}$. Fix $x \in X$. Then there exists a strictly-increasing sequence $(r_n)_n$ of natural numbers and a sequence of net elements $(N_n)_n \subseteq \mathcal{N}$ satisfying $x \in N_n$,

$\operatorname{diam}(N_n) \leq 2^{-r_n}$, and $K^{\mathscr{R}}(N_n) \leq s \cdot r_n$ for all $n \in \omega$. Then, $(i_n)_n := (\iota^{-1}(N_n))_n \in \mathcal{R}(x)$ is a $\mathcal{N}$-name of x satisfying,

$$-\frac{K^{\mathscr{R}}(i_n)}{\log \operatorname{diam}(i_n)} \leq \frac{s \cdot r_n}{r_n} = s,$$

where the left-hand side is defined as 0 when $\operatorname{diam}(i_n) = 0$. Thus, by the Coding Theorem 1.8.2 relativized to $\mathscr{R}$, we have $\dim_{\text{loc}} \mathbf{m}(x) \leq s$. As $x \in X$ was arbitrary, $\dim_{\text{loc}} \mathbf{m}(X) \leq s$.

In the other direction, suppose $\dim_{\text{loc}} \mathbf{m}(X) < s' < s \in \mathbb{Q}$. Fix $x \in X$. Then, $\dim_{\text{loc}} \mathbf{m}(x) < s'$ as well. Again by the Coding Theorem 1.8.2 relativized to $\mathscr{R}$, there exists an $\mathcal{N}$-name $(i_r)_r \in \mathcal{R}(x)$ satisfying for each $r \in \omega$:

$$-\frac{K^{\mathscr{R}}(i_r)}{\log \operatorname{diam}(i_r)} \leq s.$$

Passing to a subsequence $(r_n)_n$ such that $\operatorname{diam}(i_{r_n}) \leq 2^{-n}$, which is possible by $\operatorname{diam}(i_r) \to 0$, we see:

$$\underline{\gamma}(x) \leq \liminf_{n\to\infty} \frac{K(i_{r_n})}{-\log \operatorname{diam}(i_{r_n})} \leq s.$$

Thus, $\underline{\gamma}(X) \leq s$ as well. □

4.5.4 Effective Hausdorff Dimension

Let us now present the effectivized Hausdorff approach for effective dimension over a mesh space. See [46] for a comparison to the same notion for metric spaces with computable nice covers, or [41, 52] for that on Cantor space.

We start by generalizing two of the weights found in Section 1.5 to collections of mesh elements.

Definition 4.5.7. Fix a premeasure ρ and a mesh $\mathcal{N}$ on Ω. If $\mathcal{V}$ is a collection of indices of elements in $\mathcal{N}$, define the

- *Direct ρ-weight* of $\mathcal{V}$: $\mathrm{DW}_\rho(\mathcal{V}) := \sum_{i\in\mathcal{V}} \rho(\iota(i))$, and
- *Prefix ρ-weight* of $\mathcal{V}$: $\mathrm{PW}_\rho(\mathcal{V}) := \sup\{\mathrm{DW}_\rho(\mathcal{E}) : \mathcal{E} \sqsubseteq_{\mathrm{PF}} \mathcal{V}\}$.

We may similarly extend some of the test notions as to those defined in Section 1.5 but now for mesh spaces. As before, we restrict the premeasure to be upper-semicomputable in the fixed ω-presentation $\mathscr{R}$ of $\mathcal{N}$.

Definition 4.5.8. Fix an $\mathscr{R}$-upper-semicomputable premeasure ρ on the mesh $\mathcal{N}$. Suppose $\mathcal{U} = (U_n)_{n\in\omega}$ denotes a uniformly-$\mathscr{R}$-c.e. sequence of collections of (indices of) mesh elements from $\mathcal{N}$. Then, define $\mathcal{U}$ to be:

- a *Martin-Löf-$\mathscr{R}$-ρ-test* if $\mathrm{DW}_\rho(U_n) \leq 2^{-n}$ for all $n \in \omega$; and
- a *strong Martin-Löf-$\mathscr{R}$-ρ-test*: if $\mathrm{PW}_\rho(U_n) \leq 2^{-n}$ for all $n \in \omega$.

And suppose $\mathcal{V}$ denotes an $\mathscr{R}$-c.e. collection of (indices of) mesh elements from $\mathcal{N}$. Then, define $\mathcal{V}$ to be:

- a *Solovay-$\mathscr{R}$-ρ-test* if $\mathrm{DW}_\rho(\mathcal{V}) < +\infty$; and
- a *strong Solovay-$\mathscr{R}$-ρ-test* if $\mathcal{V}$ if $\mathrm{PW}_\rho(\mathcal{V}) < +\infty$.

Similarly to the case of Cantor space, we say that a point $x \in \Omega$ is *covered* by a test if x has an $\mathcal{N}$-name which is covered by the test.

Definition 4.5.9. For any $\mathscr{R}$-upper-semicomputable premeasure ρ on the mesh $\mathcal{N}$ and any subset $X \subseteq \Omega$,

- Let $\mathcal{U} = (U_n)_{n\in\omega}$ be a (strong) ML-$\mathscr{R}$-ρ-test. Then $\mathcal{U}$ is said to *cover* X if for each $x \in X$, there exists an $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ of x such that $i_n \in \bigcap_n U_n$. Otherwise, X is said to have *passed* $\mathcal{U}$.
- Let $\mathcal{V}$ be a (strong) Solovay-$\mathscr{R}$-ρ-test. Then $\mathcal{V}$ is said to *cover* X if for all $x \in X$, there exists an $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ of x such that $i_r \in \mathcal{V}$ for infinitely many $r \in \omega$. Otherwise, X is said to have *passed* $\mathcal{V}$.

A point $x \in X$ is *ML-$\mathscr{R}$-ρ-random* if no ML-$\mathscr{R}$-ρ-test covers x. Analogous definitions apply for all ML-type and Solovay-type tests. And as is the case over Cantor space, there exists a universal ML-s-test for each left-c.e. $s > 0$.

Proposition 4.5.10. *For each left-c.e. $s > 0$, there exists a universal ML-$\mathscr{R}$-s-test. That is, there exists an ML-$\mathscr{R}$-s-test $\mathcal{U}^s = \{U^s_n\}_n$ such that for any $X \subseteq \Omega$,*

$$X \text{ is covered by some ML-}\mathscr{R}\text{-}s\text{-test} \iff \mathcal{U}^s \text{ covers } X.$$

Proof. It is possible to compute from $\mathscr{R}$ an enumeration $(\mathcal{U}^{(k)})_k$ of all ML-$\mathscr{R}$-s-tests. This essentially follows from s-m-n Theorem (see Theorem 1.5.5 of [60]). Intuitively, an enumeration of all $\mathscr{R}$-c.e. sets can be converted into an enumeration of all ML-$\mathscr{R}$-s-tests by only enumerating those strings into the test which keep the direct s-weight of the corresponding layer in the test below the required bound.

For each n, define

$$U_n^s := \bigcup_k U_{n+k+1}^{(k)}.$$

It is clear that $\mathcal{U}^s := (U_n^s)_n$ is a uniformly $\mathscr{R}$-c.e. sequence. It further holds that $\mathcal{U}^s$ is an ML-$\mathscr{R}$-s-test since for each $n \in \omega$:

$$\mathrm{DW}_s(U_n^s) \leq \sum_{k=0}^{\infty} \mathrm{DW}_s(U_{n+k+1}^{(k)}) \leq \sum_{k=0}^{\infty} 2^{-(n+k+1)} \leq 2^{-n}.$$

It is straightforward to verify that $\mathcal{U}^s$ is indeed universal among all ML-$\mathscr{R}$-s-tests. □

Definition 4.5.11. The *effective Hausdorff dimension (with respect to* $(\Omega, \mathcal{N}, \mathscr{R})$*)* of a subset $X \subseteq \Omega$ is defined as:

$$\mathrm{effdim}(X) = \inf \{s \geq 0 : \mathcal{U}^s \text{ covers } X\}.$$

For a singular point $x \in \Omega$, define its *effective Hausdorff dimension (with respect to* $(\Omega, \mathcal{N}, \mathscr{R})$*)* as:

$$\mathrm{effdim}(x) := \inf \{s > 0 : x \text{ is not ML-}\mathscr{R}\text{-}s\text{-random}\}.$$

We may use the existence of a universal ML-$\mathscr{R}$-s-test to establish the Pointwise Stability of effective Hausdorff dimension restricted to a mesh.

Proposition 4.5.12 (Pointwise Stability)**.** *For any represented mesh space* $(\Omega, \mathcal{N}, \mathscr{R})$ *and* $X \subseteq \Omega$*,*

$$\mathrm{effdim}(X) = \sup_{x \in X} \mathrm{effdim}(x).$$

Proof. That $\mathrm{effdim}(X) \geq \sup_{x \in X} \mathrm{effdim}(x)$ is clear since any ML-$\mathscr{R}$-s-test covering X will also cover $\{x\}$ for each $x \in X$.

In the reverse direction, assume $s > \sup_{x \in X} \mathrm{effdim}(x)$. Then, each $x \in X$ is covered by some ML-$\mathscr{R}$-s-test. In particular, each $x \in X$ is covered by the universal ML-$\mathscr{R}$-s-test,

$\mathcal{U}^s$. So, $\mathcal{U}^s$ covers X, meaning $s \geq \text{effdim}(X)$. □

This immediately implies the *total stability* of effective Hausdorff dimension restricted to a mesh: i.e., if $X = \bigcup_{i \in \mathcal{I}} X_i$ for some collection of subsets $X_i \subseteq \Omega$ indexed by $\mathcal{I}$, then $\text{effdim}(X) = \sup_{i \in \mathcal{I}} \text{effdim}(X_i)$.

There is a natural identification between the effectivized Hausdorff dimension restricted to a net and the lower incompressibility ratio $\underline{\gamma}$ over a net space.

Theorem 4.5.13. *For any represented net space* $(\Omega, \mathcal{N}, \mathscr{R})$ *and subset* $X \subseteq \Omega$*,*

$$\text{effdim}(X) = \underline{\gamma}(X).$$

Proof. Let us start by showing $\text{effdim}(X) \geq \underline{\gamma}(X)$. We appeal to Theorem 4.5.5 and instead prove $\text{effdim}(X) \geq \dim_{\text{loc}} \mathbf{M}(X)$. To that end, take any rationals $s > s' > \text{effdim}(X)$, and let X being ML-$\mathscr{R}$-s-null be witnessed by some ML-$\mathscr{R}$-s-test $\mathcal{U} = (U_n)_n$. Define the map $M : \omega \to \mathbb{R}$:

$$M(i) := \frac{1}{2} \sum_n n \sum_{\substack{j \in U_n \\ \text{in}(j,i)}} \text{diam}(j)^s.$$

Claim 1: M is a lower-semicomputable continuous net semimeasure in $\mathscr{R}$.

Proof. M is $\mathscr{R}$-lower-semicomputable as each U_n is $\mathscr{R}$-c.e. and the containment relation in and diameter function diam are computable from $\mathscr{R}$. Partial sums can approximate $M(N)$ from below computably in $\mathscr{R}$. We make use of (**N4**) to check the continuous net semimeasure properties (4.2) and (4.3). Across the roots, we see

$$\sum_{\text{root}(k)} M(k) = \frac{1}{2} \sum_n n \sum_{j \in U_n} \text{diam}(j)^s \leq \frac{1}{2} \sum_n \text{DW}_s(U_n) \leq \sum_n n \cdot 2^{-(n+1)} = 1.$$

And for $i \in \omega$, the map satisfies the recursive rule:

$$\begin{aligned} M(i) &= \sum_{\text{pred}(i,j)} M(j) + \begin{cases} \frac{1}{2} n \cdot \text{diam}(i)^s & \text{if } i \in U_n \\ 0 & \text{otherwise} \end{cases} \\ &\geq \sum_{\text{pred}(i,j)} M(j). \end{aligned}$$

□

Claim 2: M s-succeeds on X.

Proof. Since $\mathcal{U}$ covers X, for each $x \in X$ and $n \in \omega$, there exists $i_n \in U_n$ such that $x \in \iota(i_n)$. Note that $\operatorname{diam}(i_n)^s \leq 2^{-n}$ since $\mathcal{U}$ is an ML-$\mathscr{R}$-s-test. Therefore, $(i_n)_n \in \mathcal{R}(x)$ is an $\mathcal{N}$-name of x. If ever $\operatorname{diam}(i_n) = 0$, then we are done. Otherwise, the name satisfies:

$$\limsup_n \frac{M(i_n)}{\operatorname{diam}(i_n)^s} \geq \limsup_n \frac{\frac{1}{2}n \cdot \operatorname{diam}(i_n)^s}{\operatorname{diam}(i_n)^s} = \limsup_n \frac{n}{2} = \infty.$$

Thus, M s-succeeds on X. □

This suffices to show that $s \geq \underline{\gamma}(x)$.

In the other direction, take some rationals $s > s' > s'' > \underline{\gamma}(X)$. There is a uniform algorithm in $\mathscr{R}$ to enumerate the following family of sets, where $n \in \omega$,

$$U_n := \left\{ i \in \omega : K^{\mathscr{R}}(i) \leq -s \cdot \log \operatorname{diam}(i) - n \right\},$$

where $\log(0)$ is defined to be $-\infty$.

We claim that f is an ML-$\mathscr{R}$-s-test covering X. First, the Kraft Inequality 1.6.4 implies that for each n,

$$\sum_{i \in U_n} \operatorname{diam}(i)^s \leq \sum_{i \in U_n} 2^{-K^{\mathscr{R}}(i) - n} \leq 2^{-n}.$$

All that remains is to show that $\mathcal{U}$ covers X. Fix $x \in X$ and $n \in \omega$. Take $r_0 \in \omega$ to be sufficiently large such that for all $r \geq r_0$,

$$(s - s')r > n.$$

Then, by $\underline{\gamma}(X) < s'' < s'$, we have that there are infinitely many $r \geq r_0$ for which $K_r^{\mathscr{R}}(x) < s'r$. For each such r, by the definition of $K_r^{\mathscr{R}}$, there exists a net element of index $i_r \in \omega$ such that $x \in \iota(i_r)$, $\operatorname{diam}(i_r) \leq 2^{-r}$, and $K^{\mathscr{R}}(i) < s'r$; thus,

$$K^{\mathscr{R}}(i_r) < s'r < sr - n \leq -s \cdot \log \operatorname{diam}(i_r) - n.$$

That is, $i_r \in U_n$. Thus, $X \subseteq \bigcap_n \bigcup_{i \in U_n} \iota(i)$. And so, $s \geq \operatorname{effdim}(X)$. □

4.5.5 Local Dimension of Outer Measures

Again inspired by the Billingsley-type definition on the local dimension of an outer measure as in Falconer [12], we define the *local net dimension* of an outer measure on a computable net space.

Definition 4.5.14. If μ is an outer measure on computable mesh space $(\Omega, d, \mathcal{N}, \mathscr{R}, \alpha)$, the *(lower) local net dimension of* $x \in \Omega$ *under* μ is:

$$\dim_{\text{loc}} \mu(x) = \inf \left\{ \liminf_{n\to\infty} \frac{\log \mu(\iota(i_n))}{\log \operatorname{diam}(i_n)} : (i_n)_n \in \mathcal{R}(x) \right\}.$$

And if $X \subseteq \Omega$, then the *(lower) local net dimension of* μ *on* X is:

$$\dim_{\text{loc}} \mu(X) = \sup_{x\in X} \dim_{\text{loc}} \mu(x).$$

Proposition 4.5.15. *For any computable grate space* $(\Omega, d, \mathcal{N}, \mathscr{R}, \alpha)$ *with layered-disjoint grate* $\mathcal{N}$*, and any subset* $X \subseteq \Omega$*,*

$$\dim_{\text{loc}} \boldsymbol{\kappa}(X) = \dim_{\text{loc}} \mathbf{m}(X).$$

Proof. This holds by a short algebraic argument. Given $N \in \mathcal{N}$, Proposition 4.4.17 implies:

$$-\log \boldsymbol{\kappa}(N) = \mathbf{K}(N) = K_{\mathcal{N}}(N) \pm o\left(-\log \operatorname{diam}_d(N)\right) = -\log \boldsymbol{\mu}(N) \pm o\left(-\log \operatorname{diam}_d(N)\right),$$

and so, for any $X \subseteq \Omega$,

$$\begin{aligned}
\dim_{\text{loc}} \boldsymbol{\kappa}(X) &= \sup_{x\in X} \inf_{(i_n)_n\in\mathcal{R}(x)} \liminf_{n\to\infty} \frac{\log \boldsymbol{\kappa}(\iota(i_n))}{\log \operatorname{diam}(i_n)} \\
&= \sup_{x\in X} \inf_{(i_n)_n\in\mathcal{R}(x)} \liminf_{n\to\infty} \frac{\log \boldsymbol{\mu}(\iota(i_n)) \pm o\left(-\log \operatorname{diam}(i_n)\right)}{\log \operatorname{diam}(i_n)} \\
&= \sup_{x\in X} \inf_{(i_n)_n\in\mathcal{R}(x)} \liminf_{n\to\infty} \frac{\log \mathbf{m}(i_n)}{\log \operatorname{diam}(i_n)} \\
&= \dim_{\text{loc}} \mathbf{m}(X).
\end{aligned}$$

□

Computable grate spaces with layered-disjoint grates thus offer the most asymptotic coincidences. In particular, over these spaces, every form of algorithmic complexity presented in this section induces the same effective dimension notion. Note that Proposition 4.5.15 holds even when $\boldsymbol{\kappa}$ is replaced by any globally or locally optimal outer measure, including $\boldsymbol{\theta}$, $\boldsymbol{\mu}$, $\mathbf{Q}$, and $\mathbf{R}$ from the previous section.

4.5.6 Summary of Asymptotic Coincidences

To summarize, we have the following theorem stating the various equalities between effective measures of information density introduced in this section over net spaces and grate spaces.

Theorem 4.5.16. *For any represented net space* $(\Omega, \mathcal{N}, \mathscr{R})$ *and any subset* $X \subseteq \Omega$*,*

$$\dim_{\text{loc}} \mathbf{M}(X) = \dim_{\text{loc}} \mathbf{m}(X) = \operatorname{cdim}(X) = \underline{\gamma}(X) = \operatorname{effdim}(X).$$

Moreover, if $(\Omega, d, \mathcal{N}, \mathscr{R}, \alpha)$ *is a computable grate space with a layered-disjoint grate, these quantities further coincide with* $\underline{\eta}(X)$ *and* $\dim_{\text{loc}} \boldsymbol{\kappa}(X)$*.*

These collected results respond to some open questions posed in [33]. In particular, we have developed Billingsley-type algorithmic dimensions over mesh spaces via mesh semimeasures, and shown how these dimensions coincide over net spaces and grate spaces. Second, they asked whether–over separable metric spaces–one could characterize algorithmic dimension via gales. In light of Proposition 4.4.18, we have indeed shown how algorithmic dimension is equivalently characterized from the perspectives of incompressibility, typicality, and unpredictability under some basic assumptions.

4.6 Point-to-Set Principles on Net Spaces

Recall the discussion of Section 1.12. It is already well known for Euclidean space that Hausdorff dimension has a pointwise, algorithmic characterization, as originally shown in 2017 by J. Lutz and N. Lutz [34]. Even more recently, P. Lutz and J. Miller succeeded at refining the Point-to-Set Principle for the s-dimensional Hausdorff outer measures [41]. We aim to establish the same point-to-set principles as in Theorem 1.12.1, Theorem 1.12.3, and Corollary 1.12.4 but for net spaces.

4.6.1 The Point-to-Set Principle for Hausdorff Dimension

The standard PTS for Hausdorff dimension holds when we restrict Hausdorff dimension to a fixed net. We call this the *Point-to-Net Principle*.

Theorem 4.6.1 (“Point-to-Net Principle” for Restricted Hausdorff Dimension)**.** *For any* *net space* $(\Omega, \mathcal{N})$ *and any* $X \subseteq \Omega$*,*

$$\dim_{\mathcal{N}}(X) = \inf_{\mathscr{R}} \sup_{x \in X} \operatorname{effdim}_{\mathscr{R}}(x),$$

where the infimum on $\mathscr{R}$ *ranges over all* ω*-presentations of* $\mathcal{N}$*.*

Proof. By Pointwise Stability 4.5.12, it suffices to check that

$$\dim_{\mathcal{N}}(X) = \inf_{\mathscr{R}} \operatorname{effdim}_{\mathscr{R}}(X).$$

It is clear that $\dim_{\mathcal{N}}(X) \leq \inf_{\mathscr{R}} \operatorname{effdim}_{\mathscr{R}}(X)$, since the left-hand side has no computability restriction on the allowed covers.

We will take $\mathscr{R}$ to be an ω-presentation of $\mathcal{N}$ powerful enough to witness $s > \dim_{\mathcal{N}}(X)$. That is, for each $\dim_{\mathcal{N}}(X) < s \in \mathbb{Q}$, there exists a sequence $(U_n^s)_{n \in \omega}$ where each $U_n^s \subseteq \omega$ satisfies $\mathrm{DW}_s(U_n^s) \leq 2^{-n}$ and $(U_n^s)_n$ covers X in the sense of an ML-type test. Then we require $\mathscr{R} \geq_{\mathrm{T}} \bigoplus \{(U_n^s)_n : \dim_{\mathcal{N}}(X) < s \in \mathbb{Q}\}$.

We claim that such an ω-presentation $\mathscr{R}$ witnesses $\dim_{\mathcal{N}}(X) = \operatorname{effdim}_{\mathscr{R}}(X)$. By definition, if $s > \dim_{\mathcal{N}}(X)$, then X is covered by $(U_n^s)_n$, which is an ML-$\mathscr{R}$-s-test. Therefore, $\operatorname{effdim}_{\mathscr{R}}(X) \leq s$ as well. So, by definition, $\operatorname{effdim}_{\mathscr{R}}(X) \leq \dim_{\mathcal{N}}(X)$. □

Moreover, if the metric space has a convenient family of nets that can compare to each s-dimensional Hausdorff premeasure, then the net dimension and Hausdorff dimension of X should agree.

Theorem 4.6.2. *Let* (Ω, d) *be a metric space, and suppose that for any* $s \geq 0$*, there exists a net* $\mathcal{N}^s$ *on* (Ω, d) *and corresponding net premeasure* ρ^s *comparable to the* s*-dimensional premeasure* ρ_s*. Then, for any* $X \subseteq \Omega$*,*

$$\dim_{\mathrm{H}} X = \dim_{\mathrm{net}}(X) = \inf_{s \geq 0} \dim_{\mathcal{N}^s}(X) = \inf_{s \geq 0} \inf_{\mathscr{R}_s} \sup_{x \in X} \operatorname{effdim}_{\mathscr{R}_s}(x),$$

where for each $s \geq 0$*, the infimum on* $\mathscr{R}_s$ *ranges over all* ω*-presentations of* $\mathcal{N}^s$*. The same conclusion holds if for any* $s \geq 0$*, there exists a net* $\mathcal{N}^s$ *such that* $\rho_s \upharpoonright \mathcal{N}_s \asymp \rho_s$*.*

In particular, if a given net $\mathcal{N}$ *satisfies* $\rho_s \upharpoonright \mathcal{N} \asymp \rho_s$ *for every* $s \geq 0$*, then for any* $X \subseteq \Omega$*:*

$$\dim_{\mathrm{H}} X = \dim_{\mathrm{net}}(X) = \dim_{\mathcal{N}}(X) = \inf_{\mathscr{R}} \sup_{x \in X} \operatorname{effdim}_{\mathscr{R}}(x),$$

where $\mathscr{R}$ *can be any* ω*-presentation of* $\mathcal{N}$*.*

Proof. It is clear that $\dim_{\mathrm{H}} X \leq \dim_{\mathrm{net}}(X)$, as the collection of covers considered for net dimension are all permitted when computing Hausdorff dimension. For the other direction, take any rational $s > \dim_{\mathrm{H}} X$. Take $\mathcal{N}^s$ and ρ^s as guaranteed for this s. By definition, $\mathcal{H}^s(X) = 0$. By the assumption that $\rho^s \asymp \rho_s$, it follows that $\mathcal{H}^{\rho_s \restriction \mathcal{N}^s}(X) = 0$, as well. So, by definition, $s \geq \dim_{\mathcal{N}^s}(X) \geq \dim_{\mathrm{net}}(X)$.

The final equality in the claim follows from the Point-to-Net Principle 4.6.1. □

Now, Theorems 4.5.16 and 4.6.1 combine to give the usual Point-to-Set Principle.

Corollary 4.6.3. *Let $(\Omega, d, \mathcal{N}, \mathscr{R}, \alpha)$ be a computable grate space where $\mathcal{N}$ is layered-disjoint and satisfies $\rho_s \restriction \mathcal{N} \asymp \rho_s$ for every $s \geq 0$. Then, for any $X \subseteq \Omega$,*

$$\dim_{\mathrm{H}} X = \min_{B \in 2^\omega} \sup_{x \in X} \dim^B(x),$$

where dim *is any of the following effective dimension notions:* $\dim_{\mathrm{loc}} \mathbf{M}$*,* $\dim_{\mathrm{loc}} \mathbf{m}$*,* $\dim_{\mathrm{loc}} \boldsymbol{\kappa}$*,* cdim*,* $\underline{\gamma}$*,* $\underline{\eta}$*, or* effdim*.*

4.6.2 Point-to-Set Principles for Hausdorff Measures

We now confirm that the point-to-set principles regarding the family of s-dimensional Hausdorff outer measures by P. Lutz and J. Miller extend to any net space $(\Omega, \mathcal{N})$.

First, though, we should extend the complexity-characterizations of partial randomness which hold over Cantor space in Theorem 1.11.2. Once again, the limiting behavior of the prefix discrepancy characterizes Martin-Löf and Solovay randomness, whereas the limiting behavior of the *a priori* discrepancy characterizes their strong forms.

Theorem 4.6.4. *Let s be left-c.e. and $(\Omega, \mathcal{N}, \mathscr{R})$ be a represented net space. Then, for any point $x \in \Omega$,*

(i) x is ML-$\mathscr{R}$-s-random if and only if $\Delta \mathrm{P}_s^{\mathscr{R}}(i_n) \geq c$,

(ii) x is Solovay-$\mathscr{R}$-s-random if and only if $\Delta \mathrm{P}_s^{\mathscr{R}}(i_n) \to +\infty$,

(iii) x is strong ML-$\mathscr{R}$-s-random if and only if $\Delta \mathrm{M}_s^{\mathscr{R}}(i_n) \geq c$,

(iv) x is strong Solovay-$\mathscr{R}$-s-random if and only if $\Delta \mathrm{M}_s^{\mathscr{R}}(i_n) \to +\infty$,

for each $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ and some constant c which may depend on $(i_n)_n$.

To begin, consider the following lemma relevant to strong Solovay randomness.

Lemma 4.6.5. *Let s be left-c.e., $\mathscr{R}$ an ω-presentation of the net $\mathcal{N}$, and $x \in \Omega$.*

(i) Suppose $\liminf_n \Delta\mathrm{M}_s^{\mathscr{R}}(i_n) \leq c$ *for some $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ and constant c. Then there exists a c.e. strong Solovay-$\mathscr{R}$-s-test $\mathcal{V}$ covering x with* $\mathrm{PW}_s(\mathcal{V}) \leq 2^c$.

(ii) Suppose $(\mathcal{V}_k)_{k\in\omega}$ is a uniformly $\mathscr{R}$-c.e. sequence of strong Solovay-$\mathscr{R}$-s-tests and $\{c_k\}_{k\in\omega}$ is an $\mathscr{R}$-computable sequence of integers such that for each k: $\mathrm{PW}_s(\mathcal{V}_k) \leq 2^{c_k}$*. Then for any x covered by all $\mathcal{V}_k$ using a common $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$, and for each $k \in \omega$,*

$$\liminf_{n\to\infty} \Delta\mathrm{M}_s^{\mathscr{R}}(i_n) \leq^+ c_k + k,$$

where the additive constant does not depend on x nor k.

Proof. (i) Define

$$\mathcal{V} := \left\{ i \in \omega : \Delta\mathrm{M}_s^{\mathscr{R}}(i) < c \right\}.$$

Clearly, $\mathcal{V}$ covering x is witnessed by the $\mathcal{N}$-name $(i_n)_n$ of x from the assumption. Now, take any $\mathcal{E} \sqsubseteq_{\mathrm{PF}} \mathcal{V}$. By the definition of $\mathcal{V}$ and by Proposition 4.3.3,

$$\mathrm{DW}_s(\mathcal{E}) = \sum_{i\in\mathcal{E}} \mathrm{diam}(i)^s \leq 2^c \sum_{i\in\mathcal{E}} 2^{-\mathrm{KM}^{\mathscr{R}}(i)} = 2^c \sum_{i\in\mathcal{E}} \mathbf{M}(i) \leq 2^c.$$

So, $\mathcal{V}$ is a strong Solovay-$\mathscr{R}$-s-test covering x with $\mathrm{PW}_s(\mathcal{V}) \leq 2^c$.

(ii) For each $k \in \omega$ define the map $M_k : \omega \to [0,1]$ (and $M : \omega \to [0,1]$) as follows:

$$M_k(i) := \frac{1}{2^{c_k+1}} \mathrm{PW}_s(\{j \in \mathcal{V}_k : \mathrm{in}(j,i)\}); \quad M(i) := \sum_{k\in\omega} \frac{M_k(i)}{2^{k+1}}.$$

We claim that each M_k–as well as M–is an $\mathscr{R}$-lower-semicomputable continuous net semimeasure. Lower-semicomputability is clear since one may approximate the prefix s-weight from below by computing direct s-weights on prefix-free subsets. The sum of any M_k across the roots of the net cannot exceed $\frac{1}{2^{c_k+1}} \mathrm{PW}_s(\mathcal{V}_k) \leq \frac{2^{c_k+1}}{2^{c_k+1}} = 1$. And for any i and $\mathcal{E} \sqsubseteq_{\mathrm{PF}} i$,

$$\begin{aligned}\sum_{e\in\mathcal{E}} M_k(e) &= \frac{1}{2^{c_k+1}} \sum_{e\in\mathcal{E}} \mathrm{PW}_s(\{j \in \mathcal{V}_k : \mathrm{in}(j,e)\}) \\ &= \frac{1}{2^{c_k+1}} \mathrm{PW}_s\left(\bigcup_{e\in\mathcal{E}} \{j \in \mathcal{V}_k : \mathrm{in}(j,e)\} \right)\end{aligned}$$

$$\leq \frac{1}{2^{c_k+1}} \operatorname{PW}_s(\{j \in \mathcal{V}_k : \operatorname{in}(j, i)\})$$
$$= M_k(i).$$

The same inequalities extend to M. Furthermore, for any $i \in \mathcal{V}_k$, we have

$$M_k(i) = \frac{1}{2^{c_k+1}} \operatorname{PW}_s(\{j \in \mathcal{V}_k : \operatorname{in}(j, i)\}) \geq \frac{1}{2^{c_k+1}} \operatorname{DW}_s(\{i\}) = \frac{1}{2^{c_k+1}} \operatorname{diam}(i)^s.$$

Let $x \in \Omega$ be covered by all $\mathcal{V}_k$ as witnessed by a fixed $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$. We conclude that for each k, there are infinitely many $n \in \omega$ for which

$$\operatorname{KM}^{\mathscr{R}}(i_n) \leq^+ -\log M(i_n) \leq -s \log \operatorname{diam}(i_n) + (c_k + 1) + (k + 1).$$

□

Proof of Theorem 4.6.4. (i) First assume that x is not ML-$\mathscr{R}$-s-random, hence covered by some ML-$\mathscr{R}$-s-test $\mathcal{U} = (U_n)_n$. For each $n \in \omega$, define the map $m_n : \omega \to \mathbb{R}$:

$$m_n(i) := \begin{cases} \frac{n}{2} \operatorname{diam}(i)^s & \text{if } i \in U_n \\ 0 & \text{otherwise,} \end{cases}$$

as well as the map $m : \omega \to \mathbb{R}$:

$$m(i) := \sum_{n \in \omega} m_n(i).$$

Each m_n is lower-semicomputable in $\mathscr{R}$ via $\mathscr{R}$-computable partial sums using the enumerations of each U_n and by approximating s from below. So too then is m lower-semicomputable in $\mathscr{R}$. Moreover, all of these maps are discrete net semimeasures, which follows from the following bound:

$$\sum_{i \in \omega} m(i) = \sum_{i \in \omega} \sum_{n \in \omega} m_n(i) = \sum_{n \in \omega} \frac{n}{2} \sum_{i \in U_n} \operatorname{diam}(i)^s = \sum_{n \in \omega} \frac{n}{2} \operatorname{DW}_s(U_n) \leq \sum_{n \in \omega} n \cdot 2^{-(n+1)} = 1.$$

By the Coding Theorem 1.8.2 relativized to $\mathscr{R}$, there exists constant C such that $m(i) \leq C \cdot 2^{-K^{\mathscr{R}}(i)}$ for almost every $i \in \omega$. Take an $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ witnessing that $\mathcal{U}$ covers x. Fixing $n \geq \lceil C \rceil + 1$,

$$K^{\mathscr{R}}(i_n) \leq -\log\left(\frac{m(i_n)}{C}\right) \leq -\log\left(\frac{n}{2C} \operatorname{diam}(i_n)^s\right) = -s \log \operatorname{diam}(i_n) - \log\left(\frac{n}{2C}\right),$$

so $\liminf_{n\to\infty} \Delta\,\mathrm{P}_s^{\mathscr{R}}(i_n) = -\infty$.

In the other direction, define for each $n \in \omega$ the $\mathscr{R}$-c.e. set:

$$U_n := \left\{ i \in \omega : \Delta\,\mathrm{P}_s^{\mathscr{R}}(i) \le -n \right\}.$$

We assume that there is some $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ for which no constant c satisfies $\Delta\,\mathrm{P}_s^{\mathscr{R}}(i_n) \ge c$. We may refine $(i_n)_n$ that to an $\mathcal{N}$-name of x such that $\Delta\,\mathrm{P}_s^{\mathscr{R}}(i_n) \le -n$. That is, $\mathcal{U} = (U_n)_n$ covers x. We confirm that $\mathcal{U}$ is an ML-$\mathscr{R}$-s-test by bounding the direct s-weights and applying the Kraft Inequality 1.6.4:

$$\mathrm{DW}_s(U_n) = \sum_{i \in U_n} \mathrm{diam}(i)^s \le \sum_{i \in U_n} 2^{-K^{\mathscr{R}}(i)-n} = 2^{-n} \sum_i 2^{-K^{\mathscr{R}}(i)} \le 2^{-n}.$$

Thus, x is not ML-$\mathscr{R}$-s-random.

(ii) Let x not being Solovay-$\mathscr{R}$-s-random be witnessed by the Solovay-$\mathscr{R}$-s-test $\mathcal{V}$ in $\mathcal{N}$. Build an $\mathscr{R}$-c.e. sequence of *axioms* $\mathcal{A} := \{\langle i, \lceil -s \cdot \log \mathrm{diam}(i) \rceil \rangle\}_{i \in \mathcal{V}}$ with

$$\sum_{i \in \mathcal{V}} 2^{-\lceil -s \cdot \log \mathrm{diam}(i) \rceil} \le \mathrm{DW}_s(\mathcal{V}) < +\infty.$$

A standard result attributed to Kraft and Chaitin implies one may demonstrate $K^{\mathscr{R}}(i) < -s \cdot \log \mathrm{diam}(i) + c$ effectively in $\mathscr{R}$ for some c and for each $i \in \mathcal{V}$. This is done by producing a prefix oracle machine which, given $\mathscr{R}$ as the oracle, produces i using a codeword of length approximately $-s \cdot \log \mathrm{diam}(i)$ for all such pairs in $\mathcal{A}$ [29]. Take $(i_n)_n \in \mathcal{R}(x)$ to be an $\mathcal{N}$-name of x witnessing that $\mathcal{V}$ covers x. Then, for infinitely many $n \in \omega$, we have $i_n \in \mathcal{V}$, so

$$\liminf_{n \to \infty} \Delta\mathrm{P}_s^{\mathscr{R}}(i_n) < c.$$

In the other direction, we assume that there is an $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ and constant $c \in \omega$ such that $\Delta\mathrm{P}_s^{\mathscr{R}}(i_n) < c$ for infinitely many $n \in \omega$. Then

$$\mathcal{V} := \left\{ i \in \omega : \Delta\mathrm{P}_s^{\mathscr{R}}(i_n) < c \right\}$$

is an $\mathscr{R}$-c.e. set which covers x via $(i_n)_n$. We confirm that $\mathcal{V}$ is a Solovay-$\mathscr{R}$-s-test using the Kraft Inequality 1.6.4:

$$\mathrm{DW}_s(\mathcal{V}) = \sum_{i \in \mathcal{V}} \operatorname{diam}(i)^s \leq \sum_{i \in \mathcal{V}} 2^{c - K^{\mathscr{R}}(i)} \leq 2^c < +\infty.$$

(iii) Suppose there is a strong ML-$\mathscr{R}$-s-test $\mathcal{U} = (U_n)_n$ covering x. Note that for each $k \in \omega$, we have $\mathcal{V}_k := \bigcup_{n>2k} U_k$ is a strong Solovay-$\mathscr{R}$-s-test covering x and with prefix s-weight:

$$\mathrm{PW}_s(\mathcal{V}_k) \leq \sum_{n>2k} \mathrm{PW}_s(U_n) \leq \sum_{n>2k} 2^{-n} = 2^{-2k}.$$

And $(\mathcal{V}_k)_k$ is uniformly $\mathscr{R}$-c.e., so by Lemma 4.6.5 (ii), there exists an $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ such that for each $k \in \omega$,

$$\liminf_n \Delta \mathrm{M}_s^{\mathscr{R}}(i_n) \leq^{+} -2k + k = -k \to -\infty.$$

In the other direction, we suppose there is an $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ for which $\liminf_n \Delta\mathrm{M}_s^{\mathscr{R}}(i_n) = -\infty$. Fixing $k \in \omega$, we have $\liminf_n \Delta\mathrm{M}_s^{\mathscr{R}}(i_n) \leq -k$, so by Lemma 4.6.5 (i), there is a strong Solovay-$\mathscr{R}$-s-test $\mathcal{V}_k$ covering x and with $\mathrm{PW}_s(\mathcal{V}_k) \leq 2^{-k}$. The proof of Lemma 4.6.5 (i) also shows that this sequence $(\mathcal{V}_k)_k$ is uniformly $\mathscr{R}$-c.e., so forms a strong ML-$\mathscr{R}$-s-test covering x.

(iv) Suppose there is a strong Solovay-$\mathscr{R}$-s-test $\mathcal{V}$ covering x. Take $c \in \mathbb{Z}$ sufficiently large so that $\mathrm{PW}_s(\mathcal{V}) \leq 2^c$. By Lemma 4.6.5 (ii), $\liminf_n \Delta\mathrm{M}_s^{\mathscr{R}}(i_n) \leq^{+} c$ for some $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$. In the other direction, if there were some c and $\mathcal{N}$-name $(i_n)_n \in \mathcal{R}(x)$ for which infinitely many $n \in \omega$ satisfy $\Delta\mathrm{M}_s^{\mathscr{R}}(i_n) \leq c$, then Lemma 4.6.5 (i) implies there exists a strong Solovay-$\mathscr{R}$-s-test $\mathcal{V}$ covering x.

□

The following lemma helps to relate the s-dimensional Hausdorff outer measure of a set restricted to a net to the existence of strong Solovay-s-tests in that net covering the set.

Lemma 4.6.6. *Let $s > 0$ be left-c.e., $\mathscr{R}$ an ω-presentation of the net $\mathcal{N}$, and $x \in \Omega$.*

(i) Suppose $\mathcal{V}$ is a strong Solovay-$\mathscr{R}$-s-test covering X. Then $(\mathcal{H} \upharpoonright \mathcal{N})^s(X) \leq \mathrm{PW}_s(\mathcal{V})$.

(ii) Suppose $(\mathcal{H} \upharpoonright \mathcal{N})^s(X) < +\infty$. Then for any $\varepsilon > 0$, there exists an oracle $B \geq_{\mathrm{T}} \mathscr{R}$ and a strong Solovay-B-s-test $\mathcal{V}$ covering X with $\mathrm{PW}_s(\mathcal{V}) \leq (\mathcal{H} \upharpoonright \mathcal{N})^s(X) + \varepsilon$.

Proof. (i) It is possible to build a sequence $\mathcal{U} = (U_n)_n$ of prefix-free subsets of $\mathcal{V}$ satisfying:

- For each $i \in U_n$, we have $\text{diam}(i) \leq 2^{-n}$; and
- For each $i \in \mathcal{V}$ of $\text{diam}(i) \leq 2^{-n}$, there exist $k \in \omega$ and $j \in U_k$ with $\text{in}(i, j)$.

Take an arbitrary $x \in X$. Notice that if $\mathcal{V}$ covers x, then there is an $\mathcal{N}$-name $(i_n)_n$ of x such that for each $k \in \omega$, there exists $n \in \omega$ such that $i_n \in U_k$. Therefore, each U_k covers X by cover elements of diameter no greater than 2^{-k}. And since $U_k \sqsubseteq_{\text{PF}} \mathcal{V}$ with $\mathcal{V}$ being a strong Solovay-$\mathscr{R}$-s-test $\mathcal{V}$, we conclude:

$$(\mathcal{H} \upharpoonright \mathcal{N})^s(X) \leq \sup_{k \in \omega} \text{DW}_s(U_k) \leq \text{PW}_s(\mathcal{V}).$$

(ii) We first build a sequence $\mathcal{U} = (U_n)_n$ subject to a few assumptions: for each $n \in \omega$,

(a) $U_n \subseteq \omega$ is contains the indices of a prefix-free colleciton of net elements;

(b) For each $i \in U_n$, we have $\text{diam}(i) \leq 2^{-n}$;

(c) $X \subseteq \bigcup_{i \in U_n} \iota(i)$;

(d) $\text{DW}_s(U_n)$ is minimal to within 2^{-n} among all other $\mathcal{U}$ satisfying the above assumptions;

(e) $\text{DW}_s(U_n)$ is minimal with respect to replacements of any collection of any indices by any net element non-trivially intersecting all of them, unless that set has diameter exceeding 2^{-n}; and

(f) For each $i \in U_{n+1}$, there exists $j \in U_n$ such that $\text{in}(i, j)$.

Given a collection $\mathcal{U}$ satisfying (a)-(e) yet with (f) failing for some smallest n and $i \in U_{n+1}$, it has to be that either i non-trivially intersects some elements of U_n or not. If so, one could replace i by these intersecting indices in U_{n+1}. Otherwise, since U_n covers X, i can be safely removed from U_{n+1} and still cover X. Either way, the problematic index can be resolved. This can be done for each U_{n+1} in succession, starting with any U_0 already satisfying (a)-(e).

Define $\mathcal{V} := \bigcup_{n > n_0} U_n$ for some sufficiently large n_0 such that $2^{-n_0} \leq \varepsilon/2$. Take $B \in 2^\omega$ to be an oracle powerful enough to both compute $\mathscr{R}$ and $\mathcal{U}$. Then $\mathcal{V}$ is certainly c.e. in B. We claim that $\mathcal{V}$ is a strong Solovay-B-s-test covering X.

Observe that it suffices to bound $\mathrm{PW}_s(\mathcal{V})$ from above by proving the same bound on $\mathrm{DW}_s(\mathcal{E})$ for any finite $\mathcal{E} \sqsubseteq_{\mathrm{PF}} \mathcal{V}$. Fixing such an $\mathcal{E}$, let $n = \max_{i \in \mathcal{E}} \lceil -\log \mathrm{diam}(i) \rceil$. Then, for each $n' > n$, we have $\mathcal{E} \cap U_{n'} = \emptyset$.

We now reduce to the case that $\mathcal{E} \subseteq U_n$ with little penalty incurred to the direct s-weight of $\mathcal{E}$. Suppose that $i \in \mathcal{E} \setminus U_n$. We replace i in $\mathcal{E}$ with by all indices $j \in U_n$ of net elements non-trivially intersecting with $\iota(i)$. By the definition of $\mathcal{V}$, any such i must have come from U_k for some $k < n$. This proposed replacement could also be performed on U_k itself. By (f), the resulting set would still cover X. Now call $\mathcal{E}'_k$ the result of doing this replacement for any violating $i \in \mathcal{E} \cap U_k \setminus U_n$, and U'_k the same but on U_k. By (d),

$$\mathrm{DW}_s(\mathcal{E}) - \mathrm{DW}_s(\mathcal{E}'_k) = \mathrm{DW}_s(U_k) - \mathrm{DW}_s(U'_k) \leq 2^{-k}.$$

So if $\mathcal{E}'$ denotes the final version of $\mathcal{E}$ after all replacements, we have:

$$\mathrm{DW}_s(\mathcal{E}) - \mathrm{DW}_s(\mathcal{E}') \leq \sum_{k=n_0+1}^{n-1} 2^{-k} < 2^{-n_0}.$$

Thus, by monotonicity,

$$\begin{aligned}\mathrm{DW}_s(\mathcal{E}) < \mathrm{DW}_s(\mathcal{E}') + 2^{-n_0} &\leq \mathrm{DW}_s(U_n) + 2^{-n_0} \\ &\leq (\mathcal{H} \upharpoonright \mathcal{N})^s(X) + 2^{-n} + 2^{-n_0} \\ &\leq (\mathcal{H} \upharpoonright \mathcal{N})^s(X) + \varepsilon.\end{aligned}$$

We conclude that $\mathrm{PW}_s(\mathcal{V}) \leq (\mathcal{H} \upharpoonright \mathcal{N})^s(X) + \varepsilon$.

□

Now we may prove the finer point-to-set principle for the s-dimensional Hausdorff outer measures restricted to a net using the limiting behavior of the *a priori* discrepancy.

Theorem 4.6.7. *For any $s \geq 0$ and $x \in \Omega$,*

$$\log(\mathcal{H} \upharpoonright \mathcal{N})^s(X) =^+ \inf_{\mathscr{R}} \sup_{x \in X} \inf_{(i_n)_n \in \mathcal{R}(x)} \liminf_{n \to \infty} [\mathrm{KM}^{\mathscr{R}}(i_n) + s \cdot \log_2 \mathrm{diam}(i_n)], \tag{4.9}$$

where the infimum on $\mathscr{R}$ ranges over all ω-presentations of $\mathcal{N}$.

Proof. Let us start by showing that $\log(\mathcal{H} \upharpoonright \mathcal{N})^s(X)$ does not exceed the right-hand side of (4.9). Fix an ω-presentation $\mathscr{R}$ of $\mathcal{N}$ and a sufficiently large integer $c \gg 1$ such that

$$\sup_{x\in X} \inf_{(i_n)_n\in\mathcal{R}(x)} \liminf_{n\to\infty} \Delta\mathrm{M}_s^{\mathscr{R}}(i_n) \leq c. \tag{4.10}$$

By Lemma 4.6.5 (i) relativized to $\mathscr{R}$, there exists a strong Solovay-$\mathscr{R}$-s-test $\mathcal{V}$ covering X with $\mathrm{PW}_s(\mathcal{V}) \leq 2^c$. And so by Lemma 4.6.6 (i), $(\mathcal{H}\restriction\mathcal{N})^s(X) \leq \mathrm{PW}_s(\mathcal{V}) \leq 2^c$.

In the other direction, start by assuming $(\mathcal{H}\restriction\mathcal{N})^s(X) \leq 2^c < +\infty$ for some integer $c \gg 1$. By Lemma 4.6.6 (ii) applied to $\varepsilon := 2^c > 0$ and an ω-presentation $\mathscr{R}_0$ of $\mathcal{N}$, there exists another ω-presentation $\mathscr{R} \geq_\mathrm{T} \mathscr{R}_0$ and a strong Solovay-$\mathscr{R}$-s-test $\mathcal{V}$ covering X with $\mathrm{PW}_s(\mathcal{V}) \leq 2^{c+1}$. We can even ask that $\mathscr{R}$ be strong enough so that: s is left-c.e. in $\mathscr{R}$, $(\mathcal{V}_k)_k$ is uniformly $\mathscr{R}$-c.e., and both of these have bounded sizes for their codes in $\mathscr{R}$. Then, by Lemma 4.6.5 (ii) relativized to $\mathscr{R}$ and applied to $\mathcal{V}_0 := \mathcal{V}$ and $c_0 := c+1$, (4.10) holds up to an additive constant independent of x, X, and s. □

One interpretation of Theorem 4.6.7 is that the map:

$$\zeta(X) := \inf_{\mathscr{R}} \sup_{x\in X} \inf_{(i_n)_n\in\mathcal{R}(x)} \liminf_{n\to\infty} \left[2^{\mathrm{KM}^{\mathscr{R}}(i_n)} \cdot \mathrm{diam}(i_n)^s\right]$$

is an outer measure on Ω commensurate with the s-dimensional Hausdorff outer measure $\rho_s \restriction \mathcal{N}$ restricted to $\mathcal{N}$. Now, under the various characterizations provided by Theorem 4.6.4, it is straightforward to derive a few other characterizations of the s-dimensional Hausdorff outer measures (restricted to a net) from Theorem 4.6.7.

Corollary 4.6.8. *For any $s \geq 0$ and $X \subseteq \Omega$,*

(i) $(\mathcal{H}\restriction\mathcal{N})^s(X) > 0$ if and only if

$$(\forall \mathscr{R} \text{ an } \omega\text{-presentation of } \mathcal{N})(\exists x \in X)\,[x \text{ is (strong) ML-}\mathscr{R}\text{-}s\text{-random}]\,.$$

(ii) $(\mathcal{H}\restriction\mathcal{N})^s(X) < +\infty$ if and only if there exists an ω-presentation $\mathscr{R}$ of $\mathcal{N}$ such that

$$\sup_{x\in X} \inf_{(i_n)_n\in\mathcal{R}(x)} \liminf_{n\to\infty} [\mathrm{KM}^{\mathscr{R}}(i_n) + s\cdot\log_2 \mathrm{diam}(i_n)] < +\infty.$$

Theorem 4.6.9. *For any $s \geq 0$ and $X \subseteq \Omega$, it holds that X is not σ-finite with respect to $(\mathcal{H}\restriction\mathcal{N})^s$ if and only if*

$$(\forall \mathscr{R} \text{ an } \omega\text{-presentation of } \mathcal{N})(\exists x \in X)\,[x \text{ is strong Solovay-}\mathscr{R}\text{-}s\text{-random}]\,.$$

Proof. We may start by assuming that there is an ω-presentation $\mathcal{N}$ for which there is no element $x \in X$ which is strong Solovay-$\mathscr{R}$-s-random. Again by the s-m-n Theorem, it is possible to uniformly-computably enumerate in $\mathscr{R}$ all of the strong Solovay-$\mathscr{R}$-s-tests $(\mathcal{V}_e)_{e\in\omega}$. By assumption, each $x \in X$ is covered by some such test, $\mathcal{V}_e$. By Lemma 4.6.6(i), for each $e \in \omega$

$$(\mathcal{H} \upharpoonright \mathcal{N})^s(\{x \in \Omega : x \text{ is covered by } \mathcal{V}_e\}) \leq \mathrm{PW}_s(\mathcal{V}_e) < +\infty.$$

Thus, X is contained in a union of sets with finite s-dimensional Hausdorff outer measure.

In the other direction, suppose X is σ-finite with respect to $(\mathcal{H} \upharpoonright \mathcal{N})^s$. Let $(X_k)_{k\in\omega}$ be a sequence witnessing this: $X \subseteq \bigcup_k X_k$, where $(\mathcal{H} \upharpoonright \mathcal{N})^s(X_k) < +\infty$ for each $k \in \omega$. By Lemma 4.6.6(ii), for each $k \in \omega$, there exists an ω-presentation $\mathscr{R}_k$ of $\mathcal{N}$ and a strong Solovay-$\mathscr{R}_k$-s-test $\mathcal{V}_k$ covering X_k. And so, letting $\mathscr{R}$ be an ω-presentation which computes all these $\mathscr{R}_k$, we have that each $x \in X$ is covered by some strong Solovay-$\mathscr{R}$-s-test. □

Similar to Theorem 4.6.2, if the metric space has a net with net premeasure commensurate with the s-dimensional Hausdorff premeasure ρ_s, all of these results will apply to the unrestricted s-dimensional Hausdorff outer measure $\mathcal{H}^s$. That is, given $s \geq 0$, suppose there exists a net $\mathcal{N}$ and a corresponding net premeasure $\rho = \Theta(\rho_s)$ on $\mathcal{N}$. (Equivalently, one could ask that $\rho_s \upharpoonright \mathcal{N} = \Theta(\rho_s)$.) Then, for any $X \subseteq \Omega$, the statements of Lemma 4.6.6, Theorem 4.6.7, Corollary 4.6.8, and Theorem 4.6.9 all apply to $\mathcal{H}^s$ in place of $(\mathcal{H} \upharpoonright \mathcal{N})^s$.

4.7 Effective Net Constructions

In [55], Rogers and Davies summarize a number of results on the existence of nets on certain classes of metric spaces, some of which further admit net measures comparable with the Hausdorff premeasures.

4.7.1 Revisiting Examples

The first example in [55] is due to A. Besicovitch.

Theorem 4.7.1 (Theorem 49 of [55])**.** *There exists a net on Euclidean space $\mathbb{R}^m$ such* *that for any dimension function h, there exists a net premeasure ρ on $\mathcal{N}$ satisfying:*

$$\mathcal{H}^{\rho_h}_\delta(X) \leq \mathcal{H}^{\rho}_\delta(X) \leq 3^m 2^{m(m+1)} \mathcal{H}^{\rho_h}_\delta(X),$$

for all $\delta \in (0,1)$ and all subsets $X \subseteq \mathbb{R}^m$. Furthermore,

$$\mathcal{H}^{\rho_h}(X) \leq \mathcal{H}^{\rho}(X) \leq 3^m 2^{m(m+1)} \mathcal{H}^{\rho_h}(X),$$

for all subsets $X \subseteq \mathbb{R}^m$.

The proof for Theorem 4.7.1 simply makes use of the prototypical dyadic net $\mathcal{Q}^m$ used in Example 4.1.8: on any $Q \in \mathcal{Q}^m_r$ with side-length 2^{-r}, define the net premeasure ρ as follows:

$$\rho(Q) := h(\mathrm{diam}_d(Q)) = h(\sqrt{m} \cdot 2^{-r}); \quad \rho(\emptyset) := 0.$$

This premeasure is exactly $\rho = \rho_h \restriction \mathcal{Q}^m$.

The second example concerns separable, ultrametric spaces.

Theorem 4.7.2 (Theorem 50 of [55])**.** *Let Ω be a separable ultrametric space with no isolated points, and let h be any dimension function. Then there is a net on Ω and corresponding net premeasure ρ such that:*

$$\mathcal{H}^{\rho_h}_\delta(X) \leq \mathcal{H}^{\rho}_\delta(X) \leq 2\mathcal{H}^{\rho_h}_{\delta/2}(X),$$

for all sufficiently small $\delta > 0$ and for all subsets $X \subseteq \Omega$. Furthermore,

$$\mathcal{H}^{\rho_h}(X) \leq \mathcal{H}^{\rho}(X) \leq 2\mathcal{H}^{\rho_h}(X),$$

for all subsets $X \subseteq \Omega$.

The proof of 4.7.2 constructs a net $\mathcal{N}$ in the same manner as in Example 4.1.10 for a fixed dimension function h based on a sequence $(d_i)_i$ converging to zero appropriately slowly for h; i.e., satisfying for all $i \in \omega$:

$$d_{i+1} > \frac{d_i}{2}, \qquad h(d_{i+1}) > \frac{h(d_i)}{2},$$

which exists because h is both non-decreasing and continuous on the right. The corresponding net premeasure is simply defined as $\rho(N) := h(\mathrm{diam}_d(N))$ for each $N \in \mathcal{N}$, or

$\rho(\emptyset) = 0$. This premeasure is exactly $\rho = \rho_h \restriction \mathcal{N}$.

4.7.2 Compact Metric Spaces

Rogers also points to a result by D. Larman in [27] on the existence of nets on "finite-dimensional," compact metric spaces. For us, Larman's notion of *finite-dimensional* means being $\mathcal{H}^s$-null for some positive integer s.

Theorem 4.7.3 (Theorem 1 of [27])**.** *Let (Ω, d) be a compact metric space. Then, there exists a Rogers net $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}^n$ being a union of layers $\mathcal{N}^n \subseteq \mathcal{P}(\Omega)$ satisfying for each $n \in \omega$:*

(i) There exists a finite collection $I^n \subset \omega^n$ of n-tuples such that $\mathcal{N}^n = \{N(\mathbf{i}) : \mathbf{i} \in I^n\}$ under an indexing $N : \bigcup_n I_n \to \mathcal{P}(\Omega)$, where $(i_1, ..., i_{n-1}, i_n) \in I^n$ implies $(i_1, ..., i_{n-1}) \in I^{n-1}$ for any $n > 1$;

(ii) For all $\mathbf{i}, \mathbf{j} \in I^n$, we have $N(\mathbf{i}) \cap N(\mathbf{j}) \neq \emptyset$ if and only if $\mathbf{i} = \mathbf{j}$;

(iii) For each $\mathbf{i} \in I^n$, $\mathrm{diam}_d(N(\mathbf{i})) < 2^{-n}$;

(iv) $\bigcup_{\mathbf{i} \in I^n} N(\mathbf{i}) = \Omega$;

(v) If $\mathbf{i} \in I^{n-1}$, then

$$\bigcup_{\mathbf{i}^\frown i_n \in I^n} N(\mathbf{i}^\frown i_n) = N(\mathbf{i});$$

(vi) Each net element of $\mathcal{N}$ is an F_σ (i.e., $\mathbf{\Sigma}_2^0$) set.

Moreover, if Ω is also $\mathcal{H}^s$-null for some positive integer s, then this net further satisfies:

(vii) For any $X \subseteq \Omega$ with $\mathrm{diam}_d(X) < 2^{-(n-1)}$ for $n > 1$, it takes at most a positive constant K many net elements from $\mathcal{N}^n$ to cover X.

Lemma 4.7.4. *Let (Ω, d) be a compact metric space and fix $r > 0$. Then, there exists a finite collection of points $\{x_i \in \Omega : i \in I\}$ which are the centers of mutually-disjoint r-balls $\{B_r(x_i) : i \in I\}$ such that their double-radius counterparts cover Ω; i.e.,*

$$\bigcup_{i \in I} B_{2r}(x_i) = \Omega.$$

Lemma 4.7.5. *If (Ω, d) is a compact metric space, then the space is $\mathcal{H}^s$-null for some positive integer s if and only if there exist positive reals K, δ, α with $\alpha < \frac{1}{2}$ such that if $d \leq \delta$, then at most K many mutually-disjoint, open balls of radius $\alpha \cdot d$ may meet any given open ball of radius d. In fact, it suffices to fix $\alpha = 2^{-7}$ and $\delta = 4$.*

Larman calls any space satisfying this latter condition a *β-space* [27].

Proof of Theorem 4.7.3. We first construct the net, which requires some setup.

For $n = 0$, put $r = r_0 = 2^{-5}$ into Lemma 4.7.4 and obtain the finite collection of points $X^0 := \{x_i : i \in I\}$ as guaranteed. Put $I^0 := I$, and consider let $x(i) := x_i$ for each $i \in I^0$. Order X^0 by following how their indices in I^0 are ordered under the standard order on natural numbers.

We inductively build an order on each successive collection X^n of the centers corresponding to level $n \in \omega$ as follows. Suppose $X^{n-1} = \{x(\mathbf{i}) : \mathbf{i} \in I^{n-1}\}$ has already been indexed by the finite collection $I^{n-1} \subset \omega^{n-1}$ and following the order induced by the lexicographic-order on I^{n-1} as a subset of ω^{n-1}. I.e., for $\mathbf{i}, \mathbf{j} \in I^{n-1}$,

$$\mathbf{i} < \mathbf{j} :\iff (\exists 1 \leq k \leq n)[i_k <_\omega j_k \text{ while } (\forall 1 \leq \ell < k)[i_\ell = j_\ell]].$$

Now for $n > 0$, substitute $r = r_n = 2^{-(n+5)}$ into Lemma 4.7.4 and obtain the finite collection of centers $X^n := \{x_i : i \in I\}$ as guaranteed. For each $\mathbf{i} \in I^{n-1}$, denote:

$$X^n(\mathbf{i}) := \left\{ x \in X^n : \begin{array}{c} (\forall \mathbf{j} \in I^{n-1})[\mathbf{j} < \mathbf{i} \implies B_{2\cdot r_n}(x) \cap B_{2\cdot r_{n-1}}(x(\mathbf{j})) = \emptyset] \wedge \\ B_{2\cdot r_n}(x) \cap B_{2\cdot r_{n-1}}(x(\mathbf{i})) \neq \emptyset \end{array} \right\}.$$

That is, collect all those level-n centers whose double-radius balls meet this double-radius ball yet do not meet any of the previous double-radius balls from level $(n-1)$. Denote by $I^n(\mathbf{i}) \subseteq I$ the indices of the centers from $X^n(\mathbf{i})$ under I. We may order $I^n(\mathbf{i})$ (and, thus, $X^n(\mathbf{i})$) by the standard ordering of natural numbers. Doing this across all $\mathbf{i} \in I^{n-1}$, we must collect all $x \in X^n$ into exactly one such $X^n(\mathbf{i})$, since the lemma guarantees the double-radius balls cover Ω. If $x = x_i \in X^n(\mathbf{i})$ for some $\mathbf{i} \in I^{n-1}$ and $i \in I$, then we let its index at level n be the n-tuple $\mathbf{i}^\frown i$, and call this $x(\mathbf{i}^\frown i) := x_i$. Collect all such indices into $I^n \subset \omega^n$ and order I^n (and, thus, X^n) lexicographically.

Just as I^n indexes the collection of centers X^n at level n, it also indexes the collection of double-radius open balls $B^n := \{B_{2\cdot r_n}(x) : x \in X^n\} = \{B(\mathbf{i}) : \mathbf{i} \in I^n\}$, where $B(\mathbf{i}) := B_{2\cdot r_n}(x(\mathbf{i}))$.

We next recall some terminology from [27].

- A *chain* is a sequence $(B_n)_{n=n_0}^{n_1}$ of balls where $n_0 \leq n_1$ are natural numbers, each $B_n \in B^n$, and $B_{n+1} \cap B_n \neq \emptyset$ for each $n = n_0, ..., n_1 - 1$.
- A point $x \in \Omega$ is *linked* to a ball $B \in B^n$ if there is a chain $(B_n)_{n=n_0}^{n_1}$ starting at $B_{n_0} = B$ and ending at some B_{n_1} containing x.
- A point $x \in \Omega$ is *strongly linked* to a ball $B \in B^n$ if x is both linked to B and not linked to any previous ball in B^n.

Now we may define the n-th layer $\mathcal{N}^n$ of the net $\mathcal{N}$ to be the collection $\{N(\mathbf{i}) : \mathbf{i} \in I^n\}$, where,

$$N(\mathbf{i}) := \{x \in \Omega : x \text{ is strongly linked to } B(\mathbf{i})\}.$$

The rest of the proof verifies the desired properties as stated in the claim.

(i) This follows directly from the construction.

(ii) No point of Ω may be strongly connected to more than one ball from B^n.

(iii) For any $x \in N(\mathbf{i})$ for some $\mathbf{i} \in I^{n_0}$, take any chain $(B_n)_{n=n_0}^{n_1}$ such that $B_{n_0} = B(\mathbf{i})$ and linking to x. Then it is easy to verify that:

$$d(x, x(\mathbf{i})) \leq 2^{-(n+2)}, \quad \text{so} \quad \operatorname{diam}(N(\mathbf{i})) < 2^{-n}.$$

(iv) By choice, $\bigcup_{\mathbf{i} \in I^n} B(\mathbf{i}) = \Omega$. So any $x \in \Omega$ is strongly linked to at least one such $B(\mathbf{i}) \in B^n$.

(v) If $\mathbf{i} \in I^{n-1}$ and $x \in N(\mathbf{i})$, then x is not linked to any ball $B(\mathbf{j})$ in B^{n-1} preceding $B(\mathbf{i})$. Then neither is x linked to any open ball $B(\mathbf{j}^\frown j_n) \in B^n$ for $\mathbf{j}^\frown j_n \in I^n$ where $\mathbf{j} < \mathbf{i}$. Yet, there is a minimal $i_n \in \omega$ satisfying $\mathbf{i}^\frown i_n \in I^n$ for which x is linked to $B(\mathbf{i}^\frown i_n) \in B^n$. Thus, $\bigcup_{\mathbf{i}^\frown i_n \in I^n} N(\mathbf{i}^\frown i_n) \supseteq N(\mathbf{i})$. Conversely, if $x \in N(\mathbf{i}^\frown i_n)$, then x is not linked to any $\mathbf{j}^\frown j_n \in I^n$ where $\mathbf{j} < \mathbf{i}$. So $x \in N(\mathbf{i})$. Thus, $\bigcup_{\mathbf{i}^\frown i_n \in I^n} N(\mathbf{i}^\frown i_n) \subseteq N(\mathbf{i})$.

(vi) For $\mathbf{i} \in I^n$, note that $N(\mathbf{i})$ is equal to the set:

$$\{x \in \Omega : x \text{ is linked to } B(\mathbf{i})\} \setminus \{x \in \Omega : x \text{ is linked to some ball in } B^n \text{ preceding } B(\mathbf{i})\}.$$

Both sets in this difference are open as the unions of open balls, so $N(\mathbf{i})$ is F_σ.

(vii) Take some $x \in \Omega$ for which $X \subseteq B_{2^{-(n-1)}}(x)$. Since (Ω, d) is compact and Ω is $\mathcal{H}^s$-null, and by the choice of the centers in X^n, Lemma 4.7.5 gives:

$$\begin{aligned} |\{\mathbf{i} \in I^n : N(\mathbf{i}) \cap X \neq \emptyset\}| &\leq |\{\mathbf{i} \in I^n : x(\mathbf{i}) \in B_{2^{-(n-2)}}(x)\}| \\ &\leq |\{B_{2^{-(n-5)}}(x(\mathbf{i})) : x(\mathbf{i}) \in B_{2^{-(n-2)}} \text{ and } \mathbf{i} \in I^n\}| \\ &\leq K. \end{aligned}$$

We claim that $\mathcal{N} := \bigcup_n \mathcal{N}^n$ is a Rogers net. The axioms (**M1**), (**M2**), and (**N4′**) are clear. And by (iii) and (iv), Ω is covered by a 2^{-n}-net of open balls for each $n \in \Omega$, so we may conclude (**M3**) as well. The containment-respecting indexing ι on $\mathcal{N}$ can be made simply by concatenating the orders from each level $\mathcal{N}^n$ as n increases. □

When Ω is both compact and $\mathcal{H}^s$-null for some positive integer s, the net constructed in Theorem 4.7.3 almost qualifies as a nice cover $(\mathcal{N}^n)_{n=1}^{\infty}$ of Ω (satisfying nice cover axioms (**A1**), (**A2**), and (**A4**)), if not for the possibility for some net elements to be of zero diameter.

Something stronger may be said for effectively compact metric spaces as defined in Section 1.4. In order to do so, we establish an effective version of the covering lemma 4.7.4.

Lemma 4.7.6. *Let (Ω, d, α) be an effectively compact metric space and fix a positive rational r. Then, there exists a finite, computable collection of centers $\{x_i : i \in I\}$ from the computable dense subset α whose r-balls $\{B_r(x_i) : i \in I\}$ are mutually-disjoint while their $3r$-balls cover Ω; i.e.,*

$$\bigcup_{i \in I} B_{3r}(x_i) = \Omega.$$

Proof. Given rational $r > 0$, let $k \in \omega$ satisfy $2^{-k} \leq r$. By effective compactness, one may compute a finite subset $\Lambda = \{x_i\}_i \subseteq \alpha$ satisfying $\Omega = \bigcup_{x_i \in \Lambda} B_{2^{-k}}(x_i)$. We build a set S by iterating through each $x_i \in \Lambda$: if x_i is not yet covered by $B_{2r}(x_j)$ for some $x_j \in S$ already, then append x_i to S. We test this by computably comparing $d(x_i, x_j) < 2r$ for each $x_j \in S$. This process is finite and computable.

It is immediate that r-balls centered at points in S are mutually-disjoint: all elements of S are at least $2r$ apart. Suppose $\bigcup_{x_j \in S} B_{3r}(x_j)$ does not cover Ω: let $x \in \Omega$ fall outside this union. By assumption, there exists $x_i \in \Lambda$ such that $x \in B_{2^k}(x_i) \subseteq B_r(x_i)$. Then, for each $x_j \in S$:

$$d(x_i, x_j) \geq d(x, x_j) - d(x, x_i) > 3r - r = 2r.$$

This contradicts the construction of S. $\square$

Corollary 4.7.7. *Every effectively compact metric space (Ω, d, α) admits a Rogers net $\mathcal{N}$ which is arithmetic and satisfies (i)-(vi) of Theorem 4.7.3. In fact, the elements of $\mathcal{N}$ are all Σ_2^0-classes. If the space is also $\mathcal{H}^s$-null for some positive integer s, then $\mathcal{N}$ further satisfies (vii) from Theorem 4.7.3.*

Proof. Larman's construction of $\mathcal{N}$ in the proof of Theorem 4.7.3 almost suffices for an effective result. We make a few modifications and discuss the consequences.

In light of the use of triple-radius balls in the effective Lemma 4.7.6, we instead define each B^n to collect triple-radius open balls $B(\mathbf{i}) := B_{3\cdot r_n}(x(\mathbf{i}))$. Moreover, it is not necessarily computable to detect whether open balls non-trivially intersect. We replace any condition of non-trivial intersection by a condition comparing the sum of the balls' radii to the distance between their centers. This affects the definition of X^n: for each $\mathbf{i} \in I^{n-1}$, denote:

$$X^n(\mathbf{i}) := \left\{ x \in X^n : \begin{array}{c} (\forall \mathbf{j} \in I^{n-1})[\mathbf{j} < \mathbf{i} \implies d(x, x(\mathbf{j})) \geq 3r_n + 3r_{n-1}] \wedge \\ d(x, x(\mathbf{i})) < 3r_n + 3r_{n-1} \end{array} \right\}.$$

Furthermore, we modify the definition of a chain to be a sequence $(B_n)_{n=n_0}^{n_1}$ of balls where $n_0 \leq n_1$, each $B_n \in B^n$, and such that their respective centers x_n and x_{n+1} have distance less than the sum of their radii:

$$d(x_n, x_{n+1}) < 3r_n + 3r_{n+1}.$$

This numerical comparison of distance versus radii is a weaker condition than checking for non-trivial intersection between their open balls, yet suffices for the construction.

Properties (i), (ii), (iv), and (v) are clear by the previous construction. We confirm that (iii) still holds under these modifications. For any $x \in N(\mathbf{i})$ with $\mathbf{i} \in I^{n_0}$, take any chain $(B_n)_{n=n_0}^{n_1}$ such that $B_{n_0} = B(\mathbf{i})$ and linking to x. It is clear by the triangle inequality that:

$$d(x, x(\mathbf{i})) \leq 3r_n + 2 \cdot 3r_n \left(2^{-1} + 2^{-2} + \cdots\right) \leq 9r_n = 9 \cdot 2^{-(n+5)} \implies \mathrm{diam}_d(N(\mathbf{i})) < 2^{-n}.$$

We note that the choice of centers forming each X^n is computable when sourced from the effective Lemma 4.7.6 and under the new definition of $X^n(\mathbf{i})$.

We next check the definability complexity of each net element as a subset of Ω. Recall that for each $\mathbf{i} \in I^n$, the set $N(\mathbf{i})$ equals:

$$\{x \in \Omega : x \text{ is linked to } B(\mathbf{i})\} \setminus \{x \in \Omega : x \text{ is linked to some ball in } B^n \text{ preceding } B(\mathbf{i})\}.$$

Denote by $\mathsf{linked}(x, \mathbf{i})$ the predicate that holds if and only if x is linked to $B(\mathbf{i})$. Then membership in $N(\mathbf{i})$ is equivalent to:

$$x \in N(\mathbf{i}) \iff \mathsf{linked}(x, \mathbf{i}) \wedge \bigwedge_{\mathbf{j}<\mathbf{i}} \neg\mathsf{linked}(x, \mathbf{j}).$$

Observe that $\mathsf{linked}(x, \mathbf{i})$ is Σ^0_1-definable: one may computably enumerate the sequences of indices of open balls forming a (finite) chain starting at $B(\mathbf{i})$ and eventually containing x. This follows by pairwise distances between elements of α being computable. Therefore, membership in $N(\mathbf{i})$ is a Σ^0_2-class.

An indexing on this net is arithmetically definable in the codes of its Σ^0_2 elements. And from the indexing, one may compute the inclusion and predecessor relations. Roots of the net are also computable using the effective lemma. Finally, implicit in the work of Moschovakis [49] (e.g., Exercise 3C.9), the diameter function is arithmetic, as expressed by the following formulas:

$$\begin{aligned} \operatorname{diam}_d(N) > \delta &\iff (\exists p, q \in N \cap \alpha)[d(p,q) > \delta], \text{ and} \\ \operatorname{diam}_d(N) < \delta &\iff \neg(\operatorname{diam}_d(N) \geq \delta) \\ &\iff \neg(\forall \varepsilon > 0)(\exists p, q \in N \cap \alpha)[d(p,q) > \delta - \varepsilon]. \end{aligned}$$

Therefore, this is an arithmetically-definable ω-presentation of $\mathcal{N}$. □

In the case of compact, $\mathcal{H}^s$-null metric spaces, Theorem 4.7.3 (vii) gives the following comparability result.

Corollary 4.7.8 (Lemma 1 of [27]). *Let Ω be a compact metric space which is $\mathcal{H}^s$-null for some positive integer s. Take $\mathcal{N}$ to be the net constructed in Theorem 4.7.3 on Ω and $K > 0$ as in (vii), and let h be a dimension function. Then, there exists a net premeasure ρ (namely, $\rho_h \restriction \mathcal{N}$) on $\mathcal{N}$ such that,*

$$\mathcal{H}^h_\delta(X) \leq \mathcal{H}^\rho_\delta(X) \leq K \cdot \mathcal{H}^h_\delta(X),$$

for all $\delta > 0$ and for all subsets $X \subseteq \Omega$. Furthermore,

$$\mathcal{H}^h(X) \le \mathcal{H}^\rho(X) \le K \cdot \mathcal{H}^h(X),$$

for all subsets $X \subseteq \Omega$.

In summary, the numerical conditions found in Theorems 4.7.1 and 4.7.2, as well as Corollary 4.7.8, help to ensure commensurateness and hence comparability between the guaranteed net premeasures and the s-dimensional Hausdorff outer measures.

4.7.3 Polish Spaces

We now perform a similar investigation for Polish spaces. A metric space is called *Polish* if it is both complete and separable, i.e. has a countable dense subset. A standard result in descriptive set theory states that each non-empty Polish spaces is the continuous image of Baire space. We are interested in the proof of this result, which utilizes Lusin schemes. Refer to [22] for more details.

Definition 4.7.9. If Ω is a set and $\mathcal{A} := (A_\sigma : \sigma \in \omega^{<\omega})$ is a family of subsets $A_\sigma \subseteq \Omega$, we call $\mathcal{A}$ a *Lusin scheme on* Ω if it satisfies for each $\sigma \in \omega^{<\omega}$ and $i, j \in \omega$:

(i) $A_{\sigma^\frown i} \cap A_{\sigma^\frown j} \neq \emptyset \iff i = j$; and

(ii) $A_{\sigma^\frown i} \subseteq A_\sigma$.

Definition 4.7.10. A Lusin scheme $\mathcal{A} = (A_\sigma)_\sigma$ on a metric space (Ω, d) is said to *have vanishing diameter* if for each $\gamma \in \omega^\omega$,

$$\lim_n \operatorname{diam}_d(A_{\gamma\restriction n}) = 0.$$

To any Lusin scheme with vanishing diameter we may associate the set $F_\mathcal{A}$ of representative addresses, as well as the representation map $f_\mathcal{A} : F_\mathcal{A} \to \Omega$, where

$$F_\mathcal{A} = \left\{\gamma \in \omega^\omega : \bigcap_n A_{\gamma\restriction n} \neq \emptyset\right\}, \quad \text{and} \quad \{f_\mathcal{A}(\gamma)\} = \bigcap_n A_{\gamma\restriction n} \quad \text{for all } \gamma \in F_\mathcal{A}.$$

The associated map $f_\mathcal{A}$ is clearly continuous and injective.

Theorem 4.7.11 (Theorem 7.9 of [22])**.** *If Ω is a Polish space, then there exists a closed class $F \subseteq \omega^\omega$ and a continuous bijection $g : F \to \Omega$. Furthermore, if $\Omega \neq \emptyset$, then there is a continuous surjection $\hat{g} : \omega^\omega \to \Omega$ extending g.*

Proof. The second claim follows from the first claim combined with the standard result in Proposition 2.8 of [22] on retractions. Fix a compatible, complete metric $d \leq 1$ on Ω. We will construct a Lusin scheme $\mathcal{A} = (A_\sigma)_\sigma$ on Ω further satisfying:

(i) $A_\emptyset = \Omega$;

(ii) Each A_σ is an F_σ (i.e., $\mathbf{\Sigma}^0_2$) class;

(iii) Each $A_\sigma = \bigcup_{i\in\omega} A_{\sigma^\frown i} = \bigcup_{i\in\omega} \overline{A_{\sigma^\frown i}}$; and

(iv) Each $\operatorname{diam}_d(A_\sigma) \leq 2^{-\operatorname{len}(\sigma)}$.

Notice that it suffices to prove that for each F_σ set $A \subseteq \Omega$ and $\varepsilon > 0$, that $A = \bigcup_{i\in\omega} A_i$, where all A_i are mutually-disjoint F_σ sets of diameter $\operatorname{diam}_d(A_i) < \varepsilon$ and whose closures satisfy $\overline{A_i} \subseteq A$.

Since A is F_σ, we may write $A = \bigcup_i C_i$, where each C_i is closed and nested in the next: $C_i \subseteq C_{i+1}$. Then, clearly,

$$A = \bigcup_i (C_{i+1} \setminus C_i),$$

giving A as a union of mutually-disjoint, $\mathbf{\Delta}^0_2$-classes. We might also cover the entire space $\Omega = \bigcup_{n\in\omega} U_n$ by open sets of bounded diameter $\operatorname{diam}_d(U_n) < \varepsilon$. Putting:

$$D^{(i)}_n := U_n \cap (C_{i+1} \setminus C_i), \quad E^{(i)}_n := D^{(i)}_n \setminus \left(\bigcup_{n'<n} D^{(i)}_{n'} \right),$$

we get that each $C_{i+1} \setminus C_i = \bigcup_n E^{(i)}_n$ and:

$$\overline{E^{(i)}_n} \subseteq \overline{C_{i+1} \setminus C_i} \subseteq \overline{C_{i+1}} = C_{i+1} \subseteq A.$$

Therefore, $A = \bigcup_{i,n} E^{(i)}_n$ is a union of mutually-disjoint, F_σ sets of diameter $\operatorname{diam}_d(E^{(i)}_n) \leq \operatorname{diam}_d(U_n) < \varepsilon$ and whose closures lie in A.

The Lusin scheme $\mathcal{A}$ indeed vanishes to zero by the assumption (iv), and thus has associated to it the set $F_\mathcal{A}$ and map $f_\mathcal{A}$ as in Definition 4.7.10. Set $F := F_\mathcal{A}$. By the definition of a Lusin scheme and by the assumptions (i) and (iii), $f_\mathcal{A}(F) = f_\mathcal{A}(F_\mathcal{A}) = A_\emptyset = \Omega$. It remains to show that F is closed.

Take $\gamma \in \omega^\omega$ and $(\gamma_k)_k \subseteq F$ such that $\gamma_k \to \gamma$. Notice that for all $\varepsilon > 0$, there exists $n_0 \in \omega$ such that $\operatorname{diam}_d(A_{\gamma\restriction n_0}) < \varepsilon$, as well as $k_0 \in \omega$ such that for all $k \geq k_0$, we have $\gamma_k \restriction n_0 = \gamma \restriction n_0$. Therefore, for each $\varepsilon > 0$ and $k, k' \geq k_0$, we have $d(f_\mathcal{A}(\gamma_k), f_\mathcal{A}(\gamma_{k'})) < \varepsilon$.

So, $(f_{\mathcal{A}}(\gamma_k))_k$ is a Cauchy sequence. But (Ω, d) is a complete metric space, so the sequence converges: $f_{\mathcal{A}}(\gamma_k) \to y \in \Omega$. But,

$$y \in \bigcap_{n\in\omega} \overline{A_{\gamma\restriction n}} = \bigcap_{n\in\omega} A_{\gamma\restriction n} \implies \gamma \in F_{\mathcal{A}} = F \quad \text{and} \quad f_{\mathcal{A}}(\gamma) = y.$$

So, F is closed, as desired. □

Corollary 4.7.12. *Every Polish space admits a Rogers net.*

Proof. Take $\mathcal{A}$ to be the Lusin scheme constructed in the proof of Theorem 4.7.11. Clearly, $\mathcal{A}$ is countable, and has the property that supersets of A_σ are just those A_ρ where ρ is a prefix of σ, of which there are only finitely many. So $\mathcal{A}$ satisfies (**M1**) and (**M2**). Furthermore, any point x in the Polish space has a preimage $\gamma \in \omega^\omega$ under the associated map $f_{\mathcal{A}}$. Notice that for each $n \in \omega$, we have $x \in A_{\gamma\restriction n}$, where $\operatorname{diam}_d(A_{\gamma\restriction n}) \leq 2^{-n}$, giving (**M3**). Now, by definition, non-empty intersection between two elements of the Lusin scheme implies comparability with respect to set-inclusion, giving (**N4′**). Note that we may index $\mathcal{A}$ by making use of a standard enumeration of $\omega^{<\omega}$ in a prefix-increasing manner. This indexing will necessarily respect set-inclusion. □

Once again, something more can be said about the definability complexity of the net for computable Polish spaces (under the assumption that they are also compact). Recall the definition of a computable Polish space from Section 1.4.

Proposition 4.7.13. *Every compact, computable Polish space (Ω, d, α) admits an arithmetically-definable Rogers net satisfying (i)-(iv) in the proof of Theorem 4.7.11. In fact, all of the elements of this net may be made to be Σ^0_2-classes.*

Proof. The construction of the Lusin scheme in the proof of Theorem 4.7.11 goes through with all sets now being of the corresponding lightface Borel complexity. One considers the recursive rule applied to any Σ^0_2-class A, beginning with $A_\emptyset = \Omega$.

The lightface construction expresses A as the countable union $\bigcup_{i,n} E^{(i)}_n$ of mutually-disjoint, uniformly-Σ^0_2-classes of diameter $\operatorname{diam}_d(E^{(i)}_n) \leq \operatorname{diam}_d(U_n) < \varepsilon$ and whose closures lie in A. By the compactness of Ω, it is arithmetic to include only those $E^{(i)}_n$ which are non-empty. And so, the Lusin scheme $\mathcal{A}$ is constructed with its arithmetically-definable indexing as well.

We have already proved that $\mathcal{A}$ qualifies as a Rogers net. The inclusion and predecessor relations are computable from the indexing. The only root of the net is $A_\emptyset = \Omega$. And, as in the proof of Corollary 4.7.7, the diameter function on the net $\mathcal{A}$ is arithmetic. So, $\mathcal{A}$ has an arithmetically-definable ω-presentation. □

Chapter 5
Applications to Geometric Measure Theory

5.1 Two Approaches to the Combinatorics of Geometric Measure Theory

We continue the discussion from Section 1.5 on (C, δ, s)-sets and their relation with Hausdorff dimension. To set up our discussion, let us recall more notation from [43]. For any natural numbers $0 < n < m$, the notation $G(m, n)$ refers to the *Grassmannian manifold* of all n-dimensional subspaces of $\mathbb{R}^m$. Whenever $V \in G(m, n)$, let $\pi_V : \mathbb{R}^m \to V$ denote to the orthogonal projection onto V. We take the usual operator norm $||\cdot||$ for linear maps as our metric on $G(m, n)$. Let $\mathcal{O}(m)$ denote the *m-dimensional orthogonal group*, which consists of all linear maps $g : \mathbb{R}^m \to \mathbb{R}^m$ preserving the inner product in $\mathbb{R}^m$. Considering the *Haar measure* θ_m on $\mathcal{O}(m)$, which is an invariant probability measure uniformly distributed on $\mathcal{O}(m)$, then $\gamma_{m,n}$ will denote the unique orthogonally-invariant, Radon probability measure on $G(m, n)$ arising from this Haar measure. That is, for any $X \subseteq G(m, n)$ and any fixed $V_0 \in X$,

$$\gamma_{m,n}(X) := \theta_m(\{g \in \mathcal{O}(m) : g(V_0) \in X\}).$$

See Section 3.9 in [43] for more details. One celebrated fact about how Hausdorff dimension behaves under orthogonal projections onto subspaces of $\mathbb{R}^m$ is commonly known as *the Marstrand-Mattila Projection Theorem*.

Theorem 5.1.1 (Marstrand-Mattila Projection Theorem; c.f. Theorem 5.8 in [44])**.** *Let*

$0 < n < m$ and $X \subseteq \mathbb{R}^m$ be analytic. Then,

$$\dim_{\mathrm{H}} \pi_V(X) = \min\{\dim_{\mathrm{H}} X, n\} \qquad \text{for } \gamma_{m,n}\text{-almost all } V \in G(m,n).$$

The $n = 1$ case of Theorem 5.1.1–as well as other basic facts about how Hausdorff dimension behaves under translations, intersections, and Cartesian products–now have alternative, effective proofs involving an application of the Point-to-Set Principle [38, 39]. Such effective proofs offer the advantage of refining what "almost all" means usually by finding a sufficiently strong notion of algorithmic randomness which witnesses the theorem.

In particular, N. Lutz and D. Stull in [39] found an effective proof that the conclusion of the $n = 1$ case of the Marstrand-Mattila Projection Theorem still holds under the alternate hypothesis that $\dim_{\mathrm{H}} X = \dim_{\mathrm{P}} X$. Shortly after, D. Stull extended this conclusion for a broader class of sets which are said to possess *optimal oracles* [66], a condition which has not yet been fully characterized in non-computability theoretic terms.

However, there was a non-computability theoretic response to the version of the Marstrand-Mattila Projection Theorem under the condition that Hausdorff and packing dimensions agree. Namely, T. Orponen in [51] employed combinatorial methods to extend the result to all $0 < n < m$ (see Theorem 1.2 in [51]).

He further proved that a weaker conclusion holds for *any* subset of $\mathbb{R}^m$.

Theorem 5.1.2 (Theorem 1.3 in [51])**.** *Let $0 < n < m$ and $X \subseteq \mathbb{R}^m$. Then*

$$\dim_{\mathrm{P}} \pi_V(X) = \min\{\dim_{\mathrm{H}} X, n\} \qquad \text{for } \gamma_{m,n}\text{-almost all } V \in G(m,n).$$

Two lemmas appear to be essential in Orponen's proofs of his Theorems 1.2 and 1.3. The first was the *Katz-Tao Lemma* (Lemma 1.5.7) claiming the existence of (C, δ, s)-sets strongly covering subsets of Hausdorff dimension less than s. The other lemma is stated below. Roughly, it claims that any bounded set appearing at precision δ to have upper box-counting dimension no more than s can be partitioned into a "bad" part which has small s-dimensional ∞-content, and a "good" part covered by some (C, δ, s)-set.

Lemma 5.1.3 (Lemma 2.3 in [51])**.** *Let $0 \le s \le m$, $\delta > 0$, $C \ge 1$, and $X \subset \mathbb{R}^m$ be a bounded subset with $N(X, \delta) \le C \cdot \delta^{-s}$. Then, for any $L \ge 1$, there exists a disjoint decomposition $X = X_{\mathrm{good}} \sqcup X_{\mathrm{bad}}$ such that*

(i) $\mathcal{H}^s_{\mathrm{I}}(X_{\mathrm{bad}}) \le C_m/L$, where C_m only depends on m, and

(ii) X_{good} *is contained in the* δ*-neighborhood of a* (CL, δ, s)*-set.*

Our main goal is to prove both these lemmas using incompressibility arguments. It is already well known that Kolmogorov complexity may be used for proving combinatorial statements, including in many of the examples from Chapter 6 of [29]. The typical advantage of an argument based on Kolmogorov complexity is in its ability to identify objects with a desired property as those which are sufficiently incompressible. And standard counting arguments show that most objects in a space should be incompressible. For simplicity, we will present these arguments over Cantor space. So let us tailor the definition of a (C, δ, s)-set to this setting.

Definition 5.1.4. Fix $m \in \omega$ and let $s > 0$, $k \in \omega$, and $C \geq 1$. A finite set $P \subseteq (2^{<\omega})^m$ is called a $(C, 2^{-k}, s)$*-set* if for any $\tau \in (2^{\leq k})^m$,

$$|P \cap [\![\tau]\!]| \leq C \cdot \left(\frac{2^{-\operatorname{len}(\tau)}}{2^{-k}}\right)^s.$$

Note that in this setting, if $P \subseteq (2^k)^m$ is a collection of m-tuple of strings all with equal length k, then $[P]$ is the $(\sqrt{m} \cdot 2^{-k})$-neighborhood about P (regarding P as the set of all its elements extended by an infinite tail of zeros).

We start with Lemma 1.5.7. The only change we have made to its statement is a small improvement in the constant term from Ck^2 to Ck.

Proposition 5.1.5 (Lemma 1.5.7, restated)**.** *Let* $0 < s \leq m$ *and let* $X \subseteq (2^\omega)^m$ *be a subset with* $\dim_{\mathrm{H}} X < s$*. Then there exists a constant* $C \geq 1$ *depending only on* m*,* s*, and* $\dim_{\mathrm{H}} X$ *such that: for every* $k \in \omega$*, there exists a* $(Ck, 2^{-k}, s)$*-set* P_k *such that the sequence* $(B_{C_m \cdot 2^{-k}}(P_k))_{k\in\omega}$ *strongly covers* X*, where* $C_m \geq 1$ *only depends on* m*.*

Proof. Let B be a Hausdorff oracle for X. Fix $C_m = \sqrt{m}$ and take $\varepsilon \in \mathbb{Q}_{>0}$ such that $\dim_{\mathrm{H}} X < s - \varepsilon$. Let us begin with a natural guess for such a strong cover. Start by defining the set of m-tuples of finite binary strings with bounded prefix complexity relative to B:

$$S := \left\{\sigma \in (2^{<\omega})^m : K^B(\sigma) < (s - \varepsilon)\operatorname{len}(\sigma) + K^B(\operatorname{len}(\sigma)) - c\right\},$$

where the constant $c > 0$ comes from item (ii) in Chaitin's Counting Theorem 1.6.5 relativized to B. For a fixed $j \in \omega$, denote the slice of S of length j:

$$S_j := S \cap (2^j)^m.$$

We claim that the sequence $([S_j])_{j\in\omega}$ must strongly cover X. The Point-to-Set Principle 1.12.1 guarantees that no $x \in X$ may have all but finitely many of its prefixes in S, or else $\dim_{\mathrm{H}} X \geq \dim^B(x) \geq s - \varepsilon$. Now, by the Counting Theorem 1.6.5,

$$|S_j| \leq 2^{(s-\varepsilon)\cdot j}.$$

Notice too that all elements in S_j are mutually at a distance at least 2^{-j}-apart, but S_j is not necessarily a $(Cj, 2^{-j}, s)$-set. Instead, for each j, we let S_j^* represent the the *s-optimal cover* of S_j in the sense of Definition 1.5.8. That is, we find an S_j^* consisting of m-tuples of strings of equal length at most j such that $[S_j] \subseteq [S_j^*]$ and S_j^* is of minimal direct s-weight. This minimum exists because we know the direct s-weight of S_j satisfies:

$$\mathrm{DW}_s(S_j) = \sum_{\sigma\in S_j} 2^{-s\cdot\mathrm{len}(\sigma)} = |S_j| \cdot 2^{-s\cdot j} \leq 2^{(s-\varepsilon)\cdot j} \cdot 2^{-s\cdot j} = 2^{-\varepsilon\cdot j}.$$

This implies our search for S_j^* should not consider strings of length shorter than $\varepsilon j/s$ (nor longer than j). Next, for fixed $j, k \in \omega$, stratify the elements of the optimal s-cover S_j^* by their string-length k:

$$S_{j,k}^* := S_j^* \cap (2^k)^m.$$

We first claim that each $S_{j,k}^*$ is a $(C_m, 2^{-k}, s)$-set. Indeed, for any $k \in \omega$ and $\tau \in (2^{\leq k})^m$,

$$2^{-s\cdot k} \cdot \left|S_{j,k}^* \cap [\![\tau]\!]\right| = \sum \left\{2^{-s\cdot\mathrm{len}(\sigma)} : \sigma \in S_{j,k}^* \wedge \sigma \succeq \tau\right\} \leq C_m \cdot 2^{-s\cdot\mathrm{len}(\tau)}.$$

Now for each $k \in \omega$, collect into $P_k := \bigcup_{j\in\omega} S_{j,k}^*$ all the strings across the $S_{j,k}^*$. We claim that each such batch P_k is a $(Ck, 2^{-k}, s)$-set, where $C = C_m s/\varepsilon$. Since $S_{j,k}^* = \emptyset$ whenever $k < \varepsilon j/s$, we may more precisely write:

$$P_k = \bigcup_{j=0}^{\lfloor ks/\varepsilon\rfloor} S_{j,k}^*.$$

So, for any $\tau \in (2^{\leq k})^m$, each $S_{j,k}^*$ being a $(C_m, 2^{-k}, s)$-set implies for any $k > 0$:

$$|P_k \cap [\![\tau]\!]| \leq \sum_{j=1}^{\lfloor ks/\varepsilon\rfloor} \left|S_{j,k}^* \cap [\![\tau]\!]\right| \leq C_m \cdot \frac{ks}{\varepsilon} \cdot \left(\frac{2^{-\mathrm{len}(\tau)}}{2^{-k}}\right)^s.$$

Finally, we show that X is indeed strongly covered by the sequence $([P_k])_{k\in\omega}$.

Fix $x \in X$. Since the $([S_j])_j$ strongly cover X, there exist infinitely many j for which $x \in [S_j] \subseteq [S_j^*]$. For each such j, there exists a corresponding $k \geq \varepsilon j/s$ such that $x \in [S_{j,k}^*] \subseteq [P_k]$. Thus, it also holds that there exist infinitely many k for which $x \in [P_k]$. □

Now we move on to Lemma 2.3 in [51].

Proposition 5.1.6 (Lemma 2.3 in [51] over Cantor Space)**.** *Let $s \in [0, m]$, $k \in \omega$, $C \geq 1$, and $X \subseteq (2^\omega)^m$ be bounded with $N(X, 2^{-k}) \leq C \cdot 2^{s \cdot k}$. Then for any $L \geq 1$, there exists a disjoint decomposition $X = X_{\text{good}} \sqcup X_{\text{bad}}$ such that*

(i) $\mathcal{H}_{\mathrm{I}}^s(X_{\text{bad}}) \leq C_m/L$, where C_m only depends on m, and

(ii) X_{good} is contained in the $(C_m \cdot 2^{-k})$-neighborhood of a $(CL, 2^{-k}, s)$-set.

Proof. Let us fix a set of "bad strings": all those strings of length at most k with an extension into X and of sufficiently small complexity,

$$S_{\text{bad}} := \left\{\sigma \in (2^{\leq k})^m : [\sigma] \cap X \neq \emptyset \quad \text{and} \quad K(\sigma) < s \cdot \operatorname{len}(\sigma) - \log L\right\}.$$

Take $X_{\text{bad}} = [S_{\text{bad}}]$ to be the open set generated by S_{bad}. Then, by the Kraft Inequality 1.6.4,

$$\mathcal{H}_{\mathrm{I}}^s(X_{\text{bad}}) \leq \mathrm{DW}_s(S_{\text{bad}}) = \sum_{\sigma \in S_{\text{bad}}} 2^{-s \cdot \operatorname{len}(\sigma)} \leq \frac{1}{L} \sum_{\sigma \in S_{\text{bad}}} 2^{-K(\sigma)} \leq \frac{1}{L}.$$

This verifies (i) in the claim. Now, put $X_{\text{good}} := X \setminus X_{\text{bad}}$, and $S_{\text{good}} := X_{\text{good}} \restriction k = \left\{\sigma \in (2^k)^m : [\sigma] \cap X_{\text{good}} \neq \emptyset\right\}$. Then,

$$X_{\text{good}} \subseteq [X_{\text{good}} \restriction k] = [S_{\text{good}}],$$

i.e., X_{good} is contained in the $(C_m \cdot 2^{-k})$-neighborhood of S_{good}.

It remains to show that S_{good} is a $(CL, 2^{-k}, s)$-set in order to conclude (ii). Fix an arbitrary $\tau \in (2^{\leq k})^m$. If $\tau \in S_{\text{bad}}$, we would have $[\tau] \subseteq X_{\text{bad}}$, so $S_{\text{good}} \cap [\![\tau]\!] = \emptyset$. Now assume that $\tau \notin S_{\text{bad}}$.

By the assumption on $N(X, 2^{-k})$, we may bound the k-precision prefix complexity of any $x \in X$ from above via a two-part description for $x \restriction k$. Fixing an enumeration of the minimal-cardinality c.e. covering of X by 2^{-k}-balls,

$$K(x \restriction k) \leq \log N(X, 2^{-k}) + O(1) \leq s \cdot k + \log C + O(1).$$

The same bound holds for any $\sigma \in X \upharpoonright k$:

$$K(\sigma) \leq s \cdot \operatorname{len}(\sigma) + \log C + O(1). \tag{5.1}$$

Suppose $\sigma \succeq \tau$ for $\sigma \in X \upharpoonright k$. Then by applying both the Coding Theorem 1.8.2 and the Counting Theorem 1.6.5 (ii) to (5.1), we deduce that τ has at most

$$2^{K(\sigma)-K(\tau)+K(k)+O(\log \operatorname{len}(\tau))+O(1)}$$

many descriptions of length $K(\sigma) + O(\log \operatorname{len}(\tau)) + O(1)$ (each $\sigma \in X \upharpoonright k$ extending τ computes $\tau = \sigma \upharpoonright \operatorname{len}(\tau)$). Now, use that $\tau \notin S_{\text{bad}}$ to bound the number of extensions of τ which are "good":

$$\begin{aligned}
|S_{\text{good}} \cap [\![\tau]\!]| &= |\{\sigma \in X_{\text{good}} \upharpoonright k : \sigma \succeq \tau\}| \\
&\leq \max\left\{2^{K(\sigma)-K(\tau)+K(k)+O(\log \operatorname{len}(\tau))+O(1)} : \sigma \in X \upharpoonright k\right\} \\
&\leq 2^{(s\cdot k+\log C+O(1))-(s\operatorname{len}(\tau)-\log L)+o(k)} \\
&= 2^{s(k-\operatorname{len}(\tau))+\log C+\log L+o(k)} \\
&\leq C' L \left(\frac{2^{-\operatorname{len}(\tau)}}{2^{-k}}\right)^{s}.
\end{aligned}$$

One could adjust the definition of S_{bad} in the beginning to absorb the extra sub-exponential terms contributing to C', resulting in a collection S_{good} which indeed is a $(CL, 2^{-k}, s)$-set. □

We discuss here how to relate the above proof with that found in [51]. The original proof forms X_{bad} as a union of all so-called *heavy* dyadic cubes, i.e., those dyadic cubes $Q \in \mathcal{Q}^m$ which contain a sufficiently many smaller sub-cubes Q' of a fixed side-length 2^{-k} intersecting with X:

$$\bigcup\left\{Q : \left|\left\{Q' : \ell(Q') = 2^{-k} \text{ and } Q' \subseteq Q \text{ and } Q' \cap X \neq \emptyset\right\}\right| \geq \alpha \cdot L \cdot \left(\frac{\ell(Q)}{2^{-k}}\right)^{s}\right\},$$

where α is C times a small constant depending on m, and $\ell(Q)$ denotes the side-length of Q. Following a strict translation between Euclidean and Cantor spaces (and omitting relativization to Hausdorff oracles), our proof could have defined X_{bad} to be the union of all infinite extensions of *heavy* strings, i.e., finite strings which may be extended to sufficiently many strings of length k whose infinite extension intersects with X:

$$\left[\left\{\sigma : |\{\tau \succeq \sigma : [\tau] \cap X \neq \emptyset \text{ and } K(\tau) \leq s \cdot \text{len}(\tau)\}| \geq \alpha \cdot L \cdot \left(\frac{2^{-\text{len}(\sigma)}}{2^{-k}}\right)^s\right\}\right].$$

The only liberty we have taken in the above translation is to add an incompressibility condition on τ, making the "bad" set even more restrictive.

In particular, any such heavy string has many *long descriptions* using each of the length k strings τ of bounded complexity extending σ and intersecting with X. Together with $\text{len}(\sigma)$, such a string τ suffices to compute σ by truncation. The Coding Theorem 1.8.2 guarantees that σ having many such long descriptions implies that σ has at least one short description. In particular, σ having at least $\alpha \cdot L \cdot 2^{s(k-\text{len}(\sigma))+K(k)}$ many descriptions $\langle \tau^*, \text{len}(\sigma)\rangle$ of length at most $K(\tau) + K(\text{len}(\sigma)) + O(1) \leq sk + o(k)$ guarantees

$$K(\sigma) \leq (s \cdot k + o(k)) - \log\left[\alpha \cdot L \cdot 2^{s(k-\text{len}(\sigma))+K(k)}\right] = s \cdot \text{len}(\sigma) - \log L - \log \alpha + o(k).$$

This essentially places σ in the canonical S_{bad} set from our proof, or its open cylinder $[\sigma]$ in the canonical X_{bad} from our proof. That is, we may view the effective analog of S_{bad} as defined by Orponen as a subset of our bad set of strings.

Finally, this result effectivizes for a restricted class of subsets. For instance,

Corollary 5.1.7. *Let $s \in [0, m] \cap \mathbb{Q}$, $k \in \omega$, $C \geq 1$, and $X \subseteq (2^\omega)^m$ be a bounded Σ^0_1-class with $N(X, 2^{-k}) \leq C \cdot 2^{s \cdot k}$. Then for any rational $L \geq 1$, there exists a disjoint decomposition $X = X_{\text{good}} \sqcup X_{\text{bad}}$ with $X_{\text{good}} \in \Pi^0_1$ and $X_{\text{bad}} \in \Sigma^0_1$ such that*

(i) $\mathcal{H}^s_{\text{I}}(X_{\text{bad}}) \leq C_m/L$, where C_m only depends on m, and

(ii) X_{good} is contained in the $(C_m \cdot 2^{-k})$-neighborhood of a $(CL, 2^{-k}, s)$-set.

5.2 Projective and Pinned-Distance Sets

In this section we present a simple effective argument which implies results for Hausdorff dimension in multiple geometric setups. These geometries will include pinned-distance sets, radial projections, and orthogonal projections.

Definition 5.2.1. Fix $0 \leq n \leq m$. Given any norm $||\cdot||_*$ on $\mathbb{R}^m$, subset $X \subseteq \mathbb{R}^m$, point $z \in \mathbb{R}^m$, and subspace $V \in G(m, n)$, denote

- The *pinned-distance* of X about z by $\Delta^*_z(X) := \{||x - z||_* : x \in X\}$;

- The *radial projection* of X about z by $P_z^*(X) := \left\{\frac{x-z}{||x-z||_*} : x \in X\right\}$;
- The *orthogonal projection* of X onto V by $\Pi_V^*(X) := \{||\mathrm{proj}_V(x)||_* : x \in X\}$;

all with respect to the norm $||\cdot||_*$. Simply write $\Delta_z(X)$, $P_z(X)$, and $\Pi_V(X)$ when taken with respect to the standard L^2-norm.

We start by recalling a result about pinned-distance sets by D. Stull [67].

Theorem 5.2.2 (Theorem 1 in [67])**.** *Let $X \subseteq \mathbb{R}^2$ be an analytic set with Hausdorff dimension strictly greater than one. Then, for all $z \in \mathbb{R}^2$ outside a set of Hausdorff dimension at most one,*

$$\dim_{\mathrm{H}}(\Delta_z X) \geq \frac{\dim_{\mathrm{H}} X}{4} + \frac{1}{2}.$$

In 2023, I. Altaf, R. Bushling, and B. Wilson proved a related lower-bound for the Hausdorff dimension of pinned-distance sets [1].

Theorem 5.2.3 (Theorem 1.2 in [1])**.** *For each $m \in \omega$, norm $||\cdot||_*$ on $\mathbb{R}^m$, point $z \in \mathbb{R}^m$, and subset $X \subseteq \mathbb{R}^m$,*

$$\dim_{\mathrm{H}}(\Delta_z^* X) \geq \dim_{\mathrm{H}} X - (m-1).$$

It is easy to see that under the conditions of Theorem 5.2.2, D. Stull's lower bound is stronger than that of Theorem 5.2.3. But the latter applies more generally: there are no exceptional directions, no dimensionality assumptions on X, nor any specificity to the L^2-norm.

Recall that the Marstrand-Mattila Projection Theorem 5.1.1 bounds from below the Hausdorff dimension of the orthogonal projection of any analytic set $X \subseteq \mathbb{R}^m$ along all but a $\gamma_{m,n}$-null set of subspaces from $G(m,n)$. Is there a lower bound for orthogonal projections which, as in Theorem 5.2.3, applies without any exceptional directions?

To answer this question, we recall a well-known bound for the Hausdorff dimension of a set which can be decomposed into two components, and provide an effective proof based on that of Theorem 5.2.3.

Theorem 5.2.4. *Let $m, n, \ell \in \omega$ and $X \subseteq \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$, and $W \subseteq \mathbb{R}^\ell$. Suppose there is a locally Lipschitz continuous map $f : U \times W \to \mathbb{R}^m$ surjective on X. Then*

$$\dim_{\mathrm{H}} X \leq \dim_{\mathrm{H}} U + \dim_{\mathrm{P}} W.$$

Classically, Theorem 5.2.4 is proved by the following sequence of inequalities:

$$\dim_{\mathrm{H}} X \leq \dim_{\mathrm{H}} f(U \times W) \leq \dim_{\mathrm{H}}(U \times W) \leq \dim_{\mathrm{H}} U + \dim_{\mathrm{P}} W.$$

The first inequality follows from monotonicity of Hausdorff dimension under inclusion. The second inequality follows from monotonicity of Hausdorff dimension under locally Lipschitz continuous mappings (see Corollary 2.4 in [12]). The final inequality originally follows from Theorem 3 in [69], which takes the most work to show.

With respect to computability, we may make use of the fact that any (locally) Lipschitz continuous map is computable in some oracle.

Proof. Let $B \geq_{\mathrm{T}} f$ be a Hausdorff oracle for both X and U, as well as a packing oracle for W. Notice that for any point $x \in X$ there exists $u \in U$ and $w \in W$ such that for any precision-level $r \in \omega$,

$$K_r^B(x) = K_r^B(f(u, w)) \leq K_r^B(u, w) + o(r) \leq K_r^B(u) + K_r^B(w) + o(r).$$

Let $\varepsilon > 0$. Since B is a Hausdorff oracle for X, there exists $x \in X$ such that $\dim_{\mathrm{H}} X \leq \dim^B(x) + \varepsilon$. Then, by the choice of B and the subadditivity of K:

$$\begin{aligned}
\dim_{\mathrm{H}} X - \varepsilon &\leq \dim^B(x) \\
&= \liminf_{r \to \infty} \frac{K_r^B(x)}{r} \\
&\leq \liminf_{r \to \infty} \frac{K_r^B(u) + K_r^B(w)}{r} \\
&\leq \dim^B(u) + \mathrm{Dim}^B(w) \\
&\leq \dim_{\mathrm{H}} U + \dim_{\mathrm{P}} W.
\end{aligned}$$

□

Let us use Theorem 5.2.4 to recreate the proof of Theorem 5.2.3.

Proof of Theorem 5.2.3. Take $U = \Delta_z^*(X)$ and $W = B_1^*(0)$ the unit ball in $\mathbb{R}^m$ with respect to the norm $||\cdot||_*$. Notice that for any $X \ni x \neq z$, we may express x as follows:

$$x = ||x - z||_* \frac{x - z}{||x - z||_*} + z.$$

Since $||x - z||_* \in \Delta_z^*(X)$ and $\frac{x-z}{||x-z||_*} \in B_1^*(0)$, we have found a mapping $f : U \times W \to \mathbb{R}^m$, $f : (u, w) \mapsto uw + z$ which surjects onto X, is locally Lipschitz continuous, and is computable in $\{z\}$ (if $x = z$, simply return z). Noting that $\dim_{\mathrm{H}} W = \dim_{\mathrm{P}} W = m - 1$, Theorem 5.2.4 gives:

$$\dim_{\mathrm{H}} X \leq \dim_{\mathrm{H}}(\Delta_z^*(X)) + (m - 1).$$

□

Proposition 5.2.5. *For each $m \in \omega$, norm $||\cdot||_*$ on $\mathbb{R}^m$, direction $e \in \mathbb{S}^{m-1}$, and subset $X \subseteq \mathbb{R}^m$,*

$$\dim_{\mathrm{H}} (\Pi_e^* X) \geq \dim_{\mathrm{H}} X - (m - 1),$$

where $\Pi_e^* = \Pi_{\mathrm{span}(e)}^*$.

Proof. Take $U = \Pi_e^*(X) \subseteq \mathbb{R}$ and $W = \mathcal{O}_e := \{w \in \mathbb{R}^m : w \perp e\} \in G(m, m-1)$ the orthogonal subspace to e. Notice that for any $x \in X$, we may express x as follows:

$$x = ||\mathrm{proj}_e(x)||_* \, e + (x - ||\mathrm{proj}_e(x)||_* \, e).$$

Since $||\mathrm{proj}_e(x)||_* \in \Pi_e^*(X)$ and $x - ||\mathrm{proj}_e(x)||_* \, e \in \mathcal{O}_e$, the mapping $f : U \times W \to \mathbb{R}^m$, $f : (u, w) \mapsto ue + w$ surjects onto X, is Lipschitz continuous, and is computable in $\{e\}$. Furthermore, $\dim_{\mathrm{H}} W = \dim_{\mathrm{P}} W = m - 1$. So, by Theorem 5.2.4,

$$\dim_{\mathrm{H}} X \leq \dim_{\mathrm{H}}(\Pi_e^*(X)) + (m - 1).$$

□

Since there is a bi-Lipschitz continuous map $||\cdot||_*$ translating between $\Pi_e^*(X)$ and $\mathrm{proj}_e(X)$, Proposition 5.2.5 also gives

$$\dim_{\mathrm{H}} (\mathrm{proj}_e(X)) \geq \dim_{\mathrm{H}} X - (m - 1).$$

We obtain another result of this kind for radial projections with a well-known corollary. We refer [50] to as a standard source.

Proposition 5.2.6. *For each $m \in \omega$, norm $||\cdot||_*$ on $\mathbb{R}^m$, point $z \in \mathbb{R}^m$, and subset*

$X \subseteq \mathbb{R}^m$,

$$\dim_{\mathrm{H}}(P_z^* X) \geq \dim_{\mathrm{H}} X - 1.$$

The proof relies on the same geometry as that of Theorem 5.2.3, but with distinct interpretations for the sets involved in the decomposition.

Proof. Take $U = P_z^*(X)$ and $W = \mathbb{R}$. Notice that for any $X \ni x \neq z$, we may express x as follows:

$$x = \frac{x - z}{||x - z||_*} \cdot ||x - z||_* + z.$$

Since $\frac{x-z}{||x-z||_*} \in P_z^*(X)$ and $||x - z||_* \in \mathbb{R}$, we have found a mapping $f : U \times W \to \mathbb{R}^m$, $f : (u, w) \mapsto uw + z$ which surjects onto X, is locally Lipschitz continuous, and is computable in $\{z\}$ (if $x = z$, simply return z). Noting that $\dim_{\mathrm{H}} W = \dim_{\mathrm{P}} W = 1$, Theorem 5.2.4 gives:

$$\dim_{\mathrm{H}} X \leq \dim_{\mathrm{H}}(P_z^*(X)) + 1.$$

□

Definition 5.2.7. Let $z \in \mathbb{R}^m$. A subset $X \subseteq \mathbb{R}^m$ is said to be *totally invisible from* z if $\dim_{\mathrm{H}}(P_z(X \setminus \{z\})) = 0$. Let $\mathrm{Inv}_{\mathrm{T}}(X)$ be the collection of all points from which X is totally invisible.

Suppose a subset X of Hausdorff dimension greater than one were to be totally invisible from some point z; then Proposition 5.2.6 would imply $\dim_{\mathrm{H}}(P_z(X \setminus \{z\})) \geq \dim_{\mathrm{H}} X - 1$, which is positive, a contradiction. This gives the well-known corollary.

Corollary 5.2.8. *For any subset* $X \subseteq \mathbb{R}^m$ *with* $\dim_{\mathrm{H}} X > 1$*, we have* $\mathrm{Inv}_{\mathrm{T}}(X) = \emptyset$.

In summary, Theorem 5.2.4 may be applied to various geometries. Theorem 5.2.3 is an application to pinned-distance sets and the unit ball. Dually, Proposition 5.2.6 is an application to radial projections and the real line. Likewise, Proposition 5.2.5 is an application to orthogonal projections and their orthogonal hyperplanes.

5.3 Subsets of Exact Measure

A standard result in geometric measure theory is one by A. Besicovitch regarding for each closed set in Euclidean space of positive $\mathcal{H}^s$-measure finding a subset whose $\mathcal{H}^s$-measure is non-zero and finite [3]. The result was quickly shown to hold more generally for analytic (or Souslin) sets by R. Davies [9]. Such a result is trivial when s equals the full dimension of the ambient space, since then $\mathcal{H}^s$ is a constant multiple of Lebesgue measure on that space. Without such a correspondence for smaller values of s, one must get around the fact that not all sets of infinite $\mathcal{H}^s$-measure are σ-finite for $\mathcal{H}^s$ (i.e., are not the countable union of subsets of finite $\mathcal{H}^s$-measure). Besicovitch circumvents this by using a net measure constructed over the net of dyadic cubes $\mathcal{Q}^m$ over Euclidean space $\mathbb{R}^m$ which is comparable to $\mathcal{H}^s$. As discussed in Chapter 4, D. Larman extended nets and net measures to generic metric spaces [27]. Soon after, C. Rogers and Davies managed to extend Besicovitch's result on the existence of finite measure subsets to compact subsets of a net space under some weak assumptions about the net and net measure [55], presented below:

Theorem 5.3.1 (Theorem 54 of [55])**.** *Let $\mathcal{N}$ be a Rogers net on a metric space (Ω, d), and ρ be a net premeasure for $\mathcal{N}$. Suppose further that*

(i) ρ is finite-valued on $\mathcal{N}$,

(ii) Each $x \in \Omega$ is $\mathcal{H}^\rho$-finite, and

(iii) If $(F_n)_{n\in\infty}$ is a decreasing sequence of compact subsets of Ω with $\mathcal{H}^\rho\left(\bigcup_n F_n\right) = 0$, it holds for each $\delta > 0$ that $\mathcal{H}^\rho_\delta(F_n) \to 0$ as $n \to \infty$.

Then, any compact set $F \subseteq \Omega$ with positive $\mathcal{H}^\rho$-measure contains a subset of positive, finite $\mathcal{H}^\rho$-measure.

Rogers and Davies used their Increasing Sets Lemma 4.2.7 and stability result 4.2.8 to prove Theorem 5.3.1.

The goal of the present section is to obtain a stronger result for a slightly more restricted class of nets and net measures. In particular, we will prove a density result which applies to any finitely-branching, layered-disjoint net with consistently shrinking diameters across the rank levels and which generates the topology of the full space. And this result will apply for any Hausdorff net premeasure (i.e., net premeasure induced by a dimension function in the sense of Section 1.3). We originally obtained the result over Cantor space by translating Besicovitch's original argument to the tree-representations of $\mathbf{\Pi}^0_1$-classes. This proof method then generalized to net spaces with the properties stated above.

Definition 5.3.2. Call a net $\mathcal{N}$ on a metric space (Ω, d) *scaling* if there exists a monotonically-decreasing function $\mathsf{d} : \omega \to (0, +\infty]$ with $\mathsf{d}(r) \to 0$ as $r \to \infty$ such that $\mathrm{rk}(N) = r$ implies $\mathrm{diam}(N) = \mathsf{d}(r)$.

Many standard examples of nets are scaling, including the net of dyadic cubes $\mathcal{Q}^m$ on $\mathbb{R}^m$ with $\mathsf{d}(r) = \sqrt{m} \cdot 2^{-r}$, or the net of cylinder sets $\mathcal{B}$ on 2^ω with $\mathsf{d}(r) = 2^{-r}$. Notice as well that the Rogers net guaranteed in Theorem 4.7.3 for a compact metric space is also scaling for $\mathsf{d}(r) = 2^{-r}$.

Definition 5.3.3. For a fixed $0 < b \in \omega$, a net $\mathcal{N}$ on a metric space (Ω, d) is *b-branching* if $\mathcal{N}$ has a single root and each net element has exactly b-many children.

Any b-branching net admits an indexing $\mathcal{N} = \{N_\sigma : \sigma \in b^{<\omega}\}$ respecting the containment relation, following the addresses in the associated graph $\mathcal{G}$ as in Proposition 4.1.5.

A collection of subsets of a topological space Ω is said to be a *subbasis of* Ω if it generates the same topology on Ω. So, a net $\mathcal{N}$ is a subbasis of (Ω, d) whenever the topology it generates on Ω is compatible with the topology induced by the metric d.

Let $\mathcal{N}$ be a scaling, b-branching, layered-disjoint net on (Ω, d) with Hausdorff net premeasure ρ_h based on a dimension function h. Suppose $F \subseteq \Omega$ is a non-empty, compact class.

As F is compact and $\mathcal{N}$ is b-branching, the net axioms (**M3**) and (**N4′**) guarantee that we may represent F as the set of paths through a tree $T_F \subseteq b^{<\omega}$. That is, it holds that any $x \in F$ may be uniquely associated with the path-wise intersection $\bigcap_{\sigma \in p} N_\sigma$, where $p \in [T_F]$ and $N_\sigma \in \mathcal{N}$ is the net element at address $\sigma \in b^{<\omega}$ in the associated graph $\mathcal{G}$.

Using T_F, we may represent the set of all non-empty, compact subclasses of F as its own $\mathbf{\Pi}^0_1$-class $\mathcal{P}_F \subseteq 2^\omega$. To do this, we define the tree $T_\mathcal{P} \subseteq 2^{<\omega}$ coding $\mathcal{P}_F$ as follows. Identify $2^{<\omega}$ with ω using a standard computable bijection with the property that for each $n \in \omega$, all strings in $b^{<\omega}$ of length n are enumerated before those of length $n + 1$. For any $\zeta \in 2^{<\omega}$, define $\zeta \in T_\mathcal{P}$ if ζ satisfies the following conditions for all $\sigma \in b^{<\omega}$ with $\zeta(\sigma) = 1$:

(i) $\zeta(\tau) = 1$ for all $\tau \preceq \sigma$,

(ii) $\sigma \in T_F$,

(iii) If ζ is defined on all immediate successors $\sigma^\frown i$ where $i \in \{0, ..., b-1\}$, then $\zeta(\sigma^\frown i) = 1$ for at least one choice of i.

It is straightforward to verify that these conditions make $T_{\mathcal{P}}$ a tree. Take any infinite path $Z \in [T_{\mathcal{P}}]$ through $T_{\mathcal{P}}$, and associate to it the set $T_Z = \{\sigma \in b^{<\omega} : Z(\sigma) = 1\}$. By condition (i) above, T_Z is a tree. By condition (ii), T_Z is a sub-tree of T_F. And by condition (iii), T_Z is pruned (i.e., each elements of T_Z has a proper extension in T_Z). Further associate to Z the compact set $E_Z \subseteq \Omega$ defined by

$$x \in E_Z \iff (\exists p \in [T_Z]) \left[\bigcap_{\sigma \in p} N_\sigma = \{x\} \right].$$

For any $Z \in [T_{\mathcal{P}}]$, since T_Z is a pruned sub-tree of T_F and F is compact, the associated set E_Z is a non-empty, compact subset of F. And any non-empty, compact subset of F may be represented by an infinite, pruned sub-tree of T_F, so we have our desired correspondence: $\mathcal{P}_F = [T_{\mathcal{P}}]$.

This representation helps to prove the following result.

Theorem 5.3.4. *Let $\mathcal{N}$ be a scaling, b-branching, layered-disjoint net and subbasis on (Ω, d) and having net Hausdorff premeasure ρ. Suppose $F \subseteq \Omega$ is compact with $(\mathcal{H} \upharpoonright \mathcal{N})^\rho(F) = +\infty$. Then there exists a compact subset $E \subseteq F$ of arbitrarily large, finite $(\mathcal{H} \upharpoonright \mathcal{N})^\rho$-measure.*

Any class F with $(\mathcal{H} \upharpoonright \mathcal{N})^\rho(F) > 0$ is certainly non-empty, so we may construct the associated class $\mathcal{P}_F$ as described above. Since 2^ω is a complete metric space and $\mathcal{P}_F \subseteq 2^\omega$ is closed, $\mathcal{P}_F$ may be viewed as a complete metric space in its own right.

Our general outline will now be as follows:

(1) Find a non-empty, compact subset $\mathcal{S} \subseteq \mathcal{P}_F$ such that if $Z \in \mathcal{S}$, then the associated subclass $E_Z \subseteq F$ satisfies $(\mathcal{H} \upharpoonright \mathcal{N})^\rho(E_Z) > 0$.

(2) Find a countable sequence of non-empty, open sets $(\mathcal{U}_n^c)_n$ in $\mathcal{P}_F$ such that if $Z \in \bigcap_{n\in\omega} \mathcal{U}_n^c$, then the associated subclass $E_Z \subseteq F$ satisfies $(\mathcal{H} \upharpoonright \mathcal{N})^\rho(E_Z) < +\infty$.

(3) Show that each $\mathcal{U}_n^c$ is dense in the induced subspace $\mathcal{S} \subseteq \mathcal{P}_F$.

Since $\mathcal{P}_F$ is complete, the closed subspace $\mathcal{S}$ in (1) will also be complete. And if each $\mathcal{U}_n^c$ is indeed dense in $\mathcal{S}$, then by the Baire Category Theorem, the intersection $\bigcap_{n\in\omega} \mathcal{U}_n^c$ will be dense in $\mathcal{S}$ as well. In particular, $\mathcal{S} \cap \bigcap_{n\in\omega} \mathcal{U}_n^c$ will be non-empty, and for any Z in this intersection, the associated subclass $E_Z \subseteq F$ will satisfy $0 < (\mathcal{H} \upharpoonright \mathcal{N})^\rho(E_Z) < +\infty$, as desired.

Suppose $U \subseteq b^{<\omega}$. Understand $\mathrm{DW}_\rho(U)$ to evaluate the direct ρ-weight of the collection of net elements $\{N_\sigma : \sigma \in U\}$ whose addresses are in U.

For any natural numbers $n \leq m$, define a function $\phi_{n,m} : \mathcal{P}_F \to \mathbb{R}$ as follows:

$$\phi_{n,m}(Z) = \min_{U \subseteq b^{<\omega}} \{\mathrm{DW}_\rho(U) : (\forall \sigma \in U)\, [n \leq \mathrm{len}(\sigma) \leq m] \text{ and } (\forall \sigma \in T_Z)(\exists \tau \in U)[\tau \preceq \sigma]\}.$$

That is, $\phi_{n,m}(Z)$ is defined as the minimum direct ρ-weight of any cover $U \subseteq b^{<\omega}$ of $[T_Z]$ by strings of length between n and m, inclusive. There are only finitely many U considered in the computation of $\phi_{n,m}(Z)$. Since $\mathcal{N}$ is scaling, any such sequence U covering $[T_Z]$ may also be viewed as giving the minimal direct ρ-weight cover of E_Z by net elements having diameter between $\mathsf{d}(m)$ and $\mathsf{d}(n)$, inclusive.

Fixing $c > 0$, define for each $n \in \omega$ the set

$$\mathcal{U}_n^c := \{Z \in \mathcal{P}_F : (\exists m \geq n)[\phi_{n,m}(Z) < c]\}.$$

For each $n \in \omega$, it only takes finitely bits of Z to determine whether $\phi_{n,m}(Z) < c$, so each $\{Z : \phi_{n,m}(Z) < c\}$ is clopen in the product topology of 2^ω. As the union of all such sets for $m \geq n$ intersected with $\mathcal{P}_F$, the set U_n^c is open in $\mathcal{P}_F$. Moreover, each U_n^c is non-empty, for if $Z \in \mathcal{P}_F$ codes the singleton set $E_Z = \{x\}$, then since $\mathcal{N}$ is scaling, we have $\phi_{n,m}(Z) = h(\mathsf{d}(m))$, as witnessed by the unique length-m string σ with $Z(\sigma) = 1$. So, pick $m \gg n$ large enough so that $h(\mathsf{d}(m)) < c$ to witness that $Z \in \mathcal{U}_n^c$.

Finally, we observe that if $Z \in \bigcap_{n\in\omega} \mathcal{U}_n^c$ with $c > 0$ being fixed, then $(\mathcal{H} \restriction \mathcal{N})^\rho(E_Z) \leq c < +\infty$. Indeed, take any $\delta > 0$, and consider $n \in \omega$ for which $\mathsf{d}(n) < \delta$. By definition of $Z \in \mathcal{U}_n^c$, there is some $m \geq n$ for which $\phi_{n,m}(Z) < c$, implying there exists a $\mathsf{d}(n)$-cover of E_Z using only elements of $\mathcal{N}$ and of direct ρ-weight less than c. Thus $(\mathcal{H} \restriction \mathcal{N})^\rho_\delta(E_Z) < c$. Taking the limit as $\delta \to 0$ gives $(\mathcal{H} \restriction \mathcal{N})^\rho(E_Z) \leq c$.

We now define the set $\mathcal{S}$ to witness being of positive $(\mathcal{H} \restriction \mathcal{N})^\rho$-measure. For each $c > 0$ and $n \in \omega$, define the set:

$$\mathcal{S}_n^c := \mathcal{P}_F \setminus \mathcal{U}_n^c = \{Z \in \mathcal{P}_F : (\forall m \geq n)[\phi_{n,m}(Z) \geq c]\}.$$

Note that each $\mathcal{S}_n^c$ is closed in $\mathcal{P}_F$ as the complement of U_n^c in $\mathcal{P}_F$.

Lemma 5.3.5. *For any $Z \in \mathcal{P}_F$, $c > 0$, and $n \in \omega$,*

$$(\mathcal{H} \restriction \mathcal{N})^\rho_{\mathsf{d}(n)}(E_Z) \geq c \iff Z \in \mathcal{S}_n^c.$$

Proof. Suppose $(\mathcal{H} \restriction \mathcal{N})^{\rho}_{\mathsf{d}(n)}(E_Z) \geq c$. Then, any $\mathsf{d}(n)$-cover of E_Z by net elements has direct ρ-weight at least c. By definition and the assumption that $\mathcal{N}$ is scaling, this includes all covers by net elements with addresses of length at least n. So, for any $m \geq n$, we have $\phi_{n,m}(Z) \geq c$, hence $Z \in S^c_n$.

In the reverse direction, assume $Z \in S^c_n$. Consider an arbitrary $\mathsf{d}(n)$-cover U of E_Z by net elements. Since E_Z is compact, it follows by $\mathcal{N}$ being a subbasis of (Ω, d) that this cover has a finite subcover $U' = \{\sigma_1, ..., \sigma_t\}$. Since $\mathcal{N}$ is scaling, each $\sigma_i \in b^{<\omega}$ is of length $\mathrm{len}(\sigma_i) \geq n$. Call m the maximum length of all such σ_i. Without loss of generality, U' may be restricted to just those σ_i with an extension in T_Z, for any σ_i without this property would not be needed in the cover of E_Z and could thus be deleted from U'.

We claim that U' is one of the covers considered in computing $\phi_{n,m}(Z)$. Indeed, since Z codes a pruned tree, any $\sigma \in b^m$ with $Z(\sigma) = 1$ must have some extension in E_Z, which in turn must have some prefix $\sigma_i \in U'$ as U' is a cover of E_Z containing strings of length at most m.

So, by the assumption that $\phi_{n,m}(Z) \geq c$, it follows that $\mathrm{DW}_\rho(U') \geq c$. And, by monotonicity, the original cover U of E_Z must also have $\mathrm{DW}_\rho(U) \geq c$. Since this $\mathsf{d}(n)$-cover by net elements was arbitrary, we may conclude $(\mathcal{H} \restriction \mathcal{N})^{\rho}_{\mathsf{d}(n)}(E_Z) \geq c$, as claimed. $\square$

By the assumption $(\mathcal{H} \restriction \mathcal{N})^{\rho}(F) = +\infty$, we have that for any $c > 0$ it must be $(\mathcal{H} \restriction \mathcal{N})^{\rho}_{\mathsf{d}(n)}(F) \geq c$ for all sufficiently large $n \gg 1$. Thus, by Lemma 5.3.5, the pruned tree representation of F itself will be an element of all the classes $\mathcal{S}^c_n$, witnessing these classes to be non-empty.

Thus, to conclude Theorem 5.3.4, it will suffice to show the following.

Proposition 5.3.6. *Let $0 < d < c$, and let $n_0 \in \omega$ be such that $\mathcal{S}^d_{n_0}$ is non-empty. Then, for all $n \in \omega$, the set $\mathcal{U}^c_n$ is dense in $\mathcal{S}^d_{n_0}$.*

This result relies on a Baire Category argument, and points to a forcing notion.

Now, before we can prove Proposition 5.3.6, let us first focus on the case when the indices of these classes agree.

Lemma 5.3.7. *Let $0 < d < c$, and let $n \in \omega$ be fixed. If $\mathcal{S}^d_n$ is non-empty, then $\mathcal{U}^c_n$ is dense in $\mathcal{S}^d_n$.*

Proof. Fix a finite binary string $\zeta \in 2^{<\omega}$ having an infinite extension $Z \in \mathcal{S}^d_n$. We claim that ζ also has an extension in $\mathcal{U}^c_n \cap \mathcal{S}^d_n$. Without loss of generality, we may assume that ζ is defined for all strings of a fixed length $n' \geq n$ by extending ζ along Z. Call k the number of length-n' strings σ for which $Z(\sigma) = 1$, and $\varepsilon := c - d > 0$. Choose $m \geq n'$ sufficiently large so that $h(\mathsf{d}(m)) < \min\{c/k, \varepsilon\}$. And define $Y_0 \in \mathcal{P}_F$ to be an infinite binary string such that for all $\sigma \in b^{<\omega}$:

(i) When $\text{len}(\sigma) \leq n'$, set $Y_0(\sigma) = Z(\sigma)$,

(ii) When $n' < \text{len}(\sigma) \leq m$, set $Y_0(\sigma) = 1$ if and only if σ is the lexicographically-least of all extensions $\nu \succeq \sigma \upharpoonright n'$ such that $Z(\nu) = 1$, and

(iii) When $\text{len}(\sigma) > m$, set $Y_0(\sigma) = 1$ if and only if both $Y_0(\sigma \upharpoonright m)$ and $Z(\sigma) = 1$.

It is straightforward to check that Y_0 satisfies all of the conditions for being in $\mathcal{P}_F$. Observe that by condition (ii), we will have exactly k strings σ of length m for which $Y_0(\sigma) = 1$, since each of the k-many length-n' strings represented in Z will yield exactly one extension. Taking U to consist of exactly these k extensions of length m, then $\text{DW}_\rho(U) = k \cdot h(\mathsf{d}(m)) < c$. This witnesses that $\phi_{n,m}(Y_0) < c$, and thus $Y_0 \in \mathcal{U}_n^c$. Note that $Y_0 \in [\zeta]$ by condition (i).

If Y_0 is also in $\mathcal{S}_n^d$, then we are done. Otherwise, let $\{\sigma_1, \ldots, \sigma_t\}$ be an enumeration of all of the strings in b^m for which $Y_0(\sigma) = 0$ yet $Z(\sigma) = 1$. At least one such string exists, or else, $Y_0 = Z$, but $Z \in \mathcal{S}_n^d$.

Inductively define the infinite binary strings $Y_1, \ldots, Y_t$ in $\mathcal{P}_F \cap [\zeta]$ as follows, where $0 \leq i < t$:

- If $Y_i(\sigma) = 1$, set $Y_{i+1}(\sigma) = 1$, or
- If $Y_i(\sigma) = 0$, set $Y_{i+1}(\sigma) = 1$ if and only if both $\sigma || \sigma_{i+1}$ and $Z(\sigma) = 1$.

Again, it is straightforward to verify that these Y_i are in fact elements of $\mathcal{P}_F \cap [\zeta]$. We also have that $Y_t = Z$, since any string $\sigma \in T_Z$ must be compatible either with something in T_{Y_0} or with one of the strings from $\{\sigma_1, \ldots, \sigma_t\}$.

We claim that Y_i must be in both $\mathcal{S}_n^d$ and $\mathcal{U}_n^c$ for some $1 \leq i \leq t$. For a fixed $m' \geq m$, consider the values of $\phi_{n,m'}(Y_0), \phi_{n,m'}(Y_1), \ldots, \phi_{n,m'}(Y_t)$. We have that $\phi_{n,m'}(Y_0) < c$, since $\phi_{n,m}(Y_0) < c$ and ϕ is monotonically-decreasing in its second coordinate. We also have that $\phi_{n,m'}(Y_t) = \phi_{n,m'}(Z) \geq d$ by assumption. Moreover, we observe for each $0 \leq i < t$ that

$$\phi_{n,m'}(Y_i) \leq \phi_{n,m'}(Y_{i+1}) < \phi_{n,m'}(Y_i) + \varepsilon.$$

The first inequality follows from the fact that $Y_i(\sigma) = 1$ implies $Y_{i+1}(\sigma) = 1$. So any valid cover for computing $\phi_{n,m'}(Y_{i+1})$, after deleting any strings not present in T_{Y_i}, gives a cover for Y_i of no greater direct ρ-weight, hence valid in computing $\phi_{n,m'}(Y_i)$. The second inequality follows from the fact that if U is the minimal cover whose direct ρ-weight witnesses $\phi_{n,m'}(Y_i)$, then adding σ_i to this cover would give a cover valid in computing $\phi_{n,m'}(Y_{i+1})$. Thus, by the choice of m,

$$\phi_{n,m'}(Y_{i+1}) \leq \phi_{n,m'}(Y_i) + h(\mathsf{d}(m)) < \phi_{n,m'}(Y_i) + \varepsilon.$$

It follows that one of the values

$$\phi_{n,m'}(Y_0), \phi_{n,m'}(Y_1), ..., \phi_{n,m'}(Y_t),$$

falls in the range $[d, d+\varepsilon) = [d, c)$, as $\phi_{n,m'}(Y_0) < c$ and $\phi_{n,m'}(Y_t) \geq d$. Let $Y^{(m')}$ denote the first element Y_i for which $\phi_{n,m'}(Y_i) \in [d, c)$. Since we only have finitely many Y_i to consider, at least one Y_i must appear as $Y^{(m')}$ for infinitely many $m' \geq m$. For such a Y_i, we have $Y_i \in \mathcal{U}_n^c$, as any of these $m' \geq m \geq n$ will work as a witness for $\phi_{n,m'}(Y_i) < c$. But since we can find arbitrarily large values of $m' \geq n$ for which $\phi_{n,m'}(Y_i) \geq d$, it follows by monotonicity that we in fact have $\phi_{n,m'}(Y_i) \geq d$ for all $m' \geq n$, giving $Y_i \in \mathcal{S}_n^d$, as desired. □

Returning to Proposition 5.3.6, by induction it now suffices to prove that if $\mathcal{U}_n^c$ is dense in $\mathcal{S}_{n_0}^d$ for some $n \geq n_0$, we also must have $\mathcal{U}_{n+1}^c$ being dense in $\mathcal{S}_{n_0}^d$. To do this, we would want to be able to pass from a cover witnessing $Z \in \mathcal{U}_n^c$ to a cover witnessing $Z \in \mathcal{U}_{n+1}^c$. Of course, the length-$n$ strings in that original cover will need to be replaced by proper extensions. The following definitions identify strings which can be replaced without significantly increasing the direct ρ-weight of the cover.

Definition 5.3.8. Let $Z \in \mathcal{P}_F$ and $\tau \in b^{<\omega}$. Then $Z^\tau \in \mathcal{P}_F$ denotes the infinite binary string defined by $Z^\tau(\sigma) = 1$ if and only if $\sigma \mid\mid \tau$ and $Z(\sigma) = 1$.

Definition 5.3.9. Let $Z \in \mathcal{P}_F$ and $n \in \omega$. Define Z to be *n-thin* if for every $\tau \in b^n$, we have

$$\lim_{m\to\infty} \phi_{n+1,m}(Z^\tau) \leq h(\mathsf{d}(n)).$$

And let $\mathcal{T}_n$ denote the set of all n-thin sequences.

As we will see, the n-thin sequences suffice for passing from $\mathcal{U}_n^c$ to $\mathcal{U}_{n+1}^c$.

Lemma 5.3.10. *For all $n \in \omega$ and $c > 0$, we have $\mathcal{U}_n^c \cap \mathcal{T}_n \subseteq \mathcal{U}_{n+1}^c$.*

Proof. Take $Z \in \mathcal{U}_n^c \cap \mathcal{T}_n$, and let $U = \{\tau_1, ..., \tau_t\}$ be a minimal cover for Z witnessing that $\phi_{n,m}(Z) < c$ for some $m \geq n$.

If all strings in U have length at least $n + 1$, then this cover also witnesses that $\phi_{n+1,m}(Z) < c$, yielding $Z \in \mathcal{U}_{n+1}^c$, as desired.

Otherwise, assume without loss of generality that there is some u with $1 \leq u \leq t$ for which $\tau_1, \tau_2, ..., \tau_u$ are exactly the strings in U of length n. Since $\mathrm{DW}_\rho(U) < c$, we may choose $\varepsilon > 0$ small enough so that we also have $\mathrm{DW}_\rho(U) < c - \varepsilon$. Since Z is n-thin, we may choose $m' \geq m$ so large that for all τ_i in U of length n, we have

$$\phi_{n+1,m'}(Z^{\tau_i}) < h(\mathsf{d}(n)) + \frac{\varepsilon}{u}.$$

For each $1 \leq i \leq t$, let V_i be a minimizing cover witnessing $\phi_{n+1,m'}(Z^{\tau_i})$, and let U' be the cover obtained by replacing each string τ_i in U with the strings from V_i. It is straightforward to verify that U' is still a valid cover for computing $\phi_{n+1,m'}(Z)$. And the direct ρ-weight of this new cover satisfies:

$$\begin{aligned}\mathrm{DW}_\rho(U') &= \sum_{1\leq i\leq u} \mathrm{DW}_\rho(V_i) + \mathrm{DW}_\rho(\{\tau_{u+1}, ..., \tau_t\}) \\ &< \sum_{1\leq i\leq u} \left(h(\mathsf{d}(n)) + \frac{\varepsilon}{u}\right) + \mathrm{DW}_\rho(\{\tau_{u+1}, ..., \tau_t\}) = \mathrm{DW}_\rho(U) + \varepsilon < c.\end{aligned}$$

This proves that Z is an element of $\mathcal{U}_{n+1}^c$, as desired. □

It remains to show that the n-thin elements are sufficiently numerous within each non-empty set $\mathcal{S}_{n_0}^d$.

Lemma 5.3.11. *Let $d > 0$ and let $n_0 \in \omega$ be such that $\mathcal{S}_{n_0}^d$ is non-empty. Then for all $n \geq n_0$, the set $\mathcal{T}_n$ is dense in $\mathcal{S}_{n_0}^d$.*

Proof. Fix $n \geq n_0$ and a finite binary string $\zeta \in 2^{<\omega}$ having an infinite extension $Z \in \mathcal{S}_{n_0}^d$. We claim that ζ has an extension in $\mathcal{T}_n \cap \mathcal{S}_{n_0}^d$. Without loss of generality, we may assume that ζ is defined for all strings of length n by extending ζ along Z. For each $\tau \in b^n$, denote by $\zeta^\tau \in 2^{\mathrm{len}(\zeta)}$ the string defined by

$$\zeta^\tau(\sigma) = 1 \iff \sigma \mid\mid \tau \text{ and } \zeta(\sigma) = 1.$$

We start by defining an element $Y \in \mathcal{T}_n \cap [\zeta]$, and later check that $Y \in \mathcal{S}_{n_0}^d$, witnessing $\mathcal{T}_n$ being dense in $\mathcal{S}_{n_0}^d$.

To define the element $Y \in \mathcal{P}_F$, it suffices to both specify Y^τ for each $\tau \in b^n$ and further ensure that each Y^τ extends ζ^τ. In this way, the final sequence Y will be well-defined.

If a string $\sigma \in b^{<\omega}$ is compatible with two distinct strings $\tau_1, \tau_2 \in b^n$, then σ must be a common prefix for both of these strings, implying $\operatorname{len}(\sigma) < n$. Therefore, $Y(\sigma)$ will be determined by $\zeta(\sigma)$ by the assumption that ζ is defined on all strings of length up to n. It is also straightforward to show that Y will be an element of $\mathcal{P}_F$ so long as each Y^τ is.

Given $\tau \in b^n$, we say that τ is *n-thin for Z* if:

$$\lim_{m\to\infty} \phi_{n+1,m}(Z^\tau) \le h(\mathsf{d}(n)).$$

We further collect into $\mathcal{T}_n^Z \subseteq b^n$ all the n-thin strings for Z. Now, define $Y^\tau = Z^\tau$ whenever $\tau \in \mathcal{T}_n^Z$. Noting that $\zeta^\tau \preceq Z^\tau$, the claim is satisfied for these τ.

Otherwise, $\tau \notin \mathcal{T}_n^Z$, so it must be that

$$\lim_{m\to\infty} \phi_{n+1,m}(Z^\tau) > h(\mathsf{d}(n)),$$

which, by monotonicity, implies that $\phi_{n+1,m}(Z^\tau) > h(\mathsf{d}(n))$ for all $m \ge n+1$, hence $Z^\tau \in \mathcal{S}_{n+1}^{h(\mathsf{d}(n))}$.

Take $(r_i)_{i\in\omega}$ to be any decreasing sequence of positive real numbers converging to 0. By Lemma 5.3.7, it follows that $\mathcal{U}_{n+1}^{h(\mathsf{d}(n))+r_i}$ is dense in $\mathcal{S}_{n+1}^{h(\mathsf{d}(n))}$ for each $i \in \omega$. And by the Baire Category Theorem, the intersection $\bigcap_i \mathcal{U}_{n+1}^{h(\mathsf{d}(n))+r_i}$ is also dense in $\mathcal{S}_{n+1}^{h(\mathsf{d}(n))}$. And since $\zeta^\tau \preceq Z^\tau \in \mathcal{S}_{n+1}^{h(\mathsf{d}(n))}$, it follows that $[\zeta^\tau] \cap \mathcal{S}_{n+1}^{h(\mathsf{d}(n))} \neq \emptyset$. Thus, we may fix a sequence $Y^\tau \in [\zeta^\tau] \cap \mathcal{S}_{n+1}^{h(\mathsf{d}(n))} \cap \bigcap_i \mathcal{U}_{n+1}^{h(\mathsf{d}(n))+r_i}$.

The fact that $Y^\tau \in \mathcal{S}_{n+1}^{h(\mathsf{d}(n))}$ implies that $\phi_{n+1,m}(Y^\tau) \ge h(\mathsf{d}(n))$ for all $m \ge n+1$. Yet, for any $\varepsilon > 0$, one may choose $i \gg 1$ for which $r_i < \varepsilon$. Then, since $Y^\tau \in \mathcal{U}_{n+1}^{h(\mathsf{d}(n))+r_i}$, there exists some $m \ge n+1$ for which

$$\phi_{n+1,m}(Y^\tau) < h(\mathsf{d}(n)) + r_i < h(\mathsf{d}(n)) + \varepsilon.$$

So, by the monotonicity of $\phi_{n+1,m}(Y^\tau)$ in its second component, we conclude that

$$\lim_{m\to\infty} \phi_{n+1,m}(Y^\tau) = h(\mathsf{d}(n)).$$

This construction guarantees that Y both extends ζ and is n-thin.

Now we show that Y is indeed an element of $\mathcal{S}_{n_0}^d$, i.e., that $\phi_{n_0,m}(Y) \ge d$ for each $m \ge n_0$. By monotonicity, it suffices to only consider $m \ge n \ge n_0$. Fix such an $m \ge n$ and suppose $U \subseteq b^{<\omega}$ is a cover valid for computing $\phi_{n_0,m}(Y)$. We wish to show that

$\mathrm{DW}_\rho(U) \geq d$.

Without loss of generality, we may take U to be prefix-free, for any strings in U having proper prefixes in U may be deleted while maintaining the cover's validity in computing $\phi_{n_0,m}(Y)$ and not increasing the direct ρ-weight of the resulting cover. We plan to push the cover U to a cover U' valid instead for computing $\phi_{n_0,m}(Z)$ and satisfying $\mathrm{DW}_\rho(U') \leq \mathrm{DW}_\rho(U)$. Since Z is an element of $\mathcal{S}^d_{n_0}$, this would yield $\mathrm{DW}_\rho(U) \geq \mathrm{DW}_\rho(U') \geq d$, putting $Y \in \mathcal{S}^d_{n_0}$, as U was arbitrary.

Consider all $\tau \in b^n \setminus \mathcal{T}^Z_n$. Denote by V_τ the collection of all proper extensions of τ in U. If $V_\tau = \emptyset$, do nothing. Otherwise, replace in the cover U all strings in V_τ by τ itself. Let us argue why this replacement will not increase the direct ρ-weight of the cover.

Note that all strings in V_τ have length between $n+1$ and m, inclusive. And if $\sigma \in b^m$ is such that $Y^\tau(\sigma) = 1$, then σ must extend some element of V_τ. Given that the original set U is a cover for Y, we have that σ extends some element $\mu \in U$ compatible with τ. If $\mu \preceq \tau$, then μ would be a proper prefix of any element in V_τ, contradicting the fact that V_τ was non-empty and U was prefix-free. Therefore, it must be that $\mu \succ \tau$, meaning $\mu \in V_\tau$. Taken together, we see that V_τ is valid cover for computing $\phi_{n+1,m}(Y^\tau)$. Since $\tau \notin \mathcal{T}^Z_n$, the construction of Y yields that $\phi_{n+1,m}(Y^\tau) \geq h(\mathsf{d}(n))$, so $\mathrm{DW}_\rho(V_\tau) \geq h(\mathsf{d}(n))$. So, as τ has length n, replacing all of V_τ with τ cannot increase the direct ρ-weight of U.

Let U' be the cover obtained by performing this replacement for all $\tau \in b^n \setminus \mathcal{T}^Z_n$. It remains to check that U' valid for computing $\phi_{n_0,m}(Z)$. Since the replacement process may only add strings of length n, and $n_0 \leq n \leq m$, the sequence U' satisfies the length condition.

For the covering condition, suppose we have a string $\sigma \in b^m$ for which $Z(\sigma) = 1$. Let $\tau = \sigma \upharpoonright n$. Suppose $\tau \in \mathcal{T}^Z_n$. Then by construction, we have $Y^\tau = Z^\tau$, so $Y^\tau(\sigma) = Z^\tau(\sigma) = 1$. As the original cover U is valid for $\phi_{n_0,m}(Y)$, this implies that U contains some prefix of σ compatible with τ. This prefix will still be in U', since we only removed strings which extended elements of $b^n \setminus \mathcal{T}^Z_n$.

Otherwise, $\tau \notin \mathcal{T}^Z_n$. The original cover U is valid for computing $\phi_{n_0,m}(Y)$, and hence valid for $\phi_{n_0,m}(Y^\tau)$, as well. Given that $\phi_{n+1,m}(Y^\tau) \geq h(\mathsf{d}(n))$, there must be at least one string $\sigma' \in b^m$ such that $Y^\tau(\sigma') = 1$. So, this σ' has a prefix $\mu \in U$ compatible with τ. If $\mu \preceq \sigma$, then μ is also a prefix of σ and is not removed when passing to the cover U'. Otherwise, μ *is* removed and replaced with τ. Either way, we get that U' contains a prefix of σ, completing the argument that U' is a valid cover for computing $\phi_{n_0,m}(Z)$, as desired. □

Now we may prove the key density result.

Proof of Proposition 5.3.6. By Lemma 5.3.7 and since the sequence $(\mathcal{U}_n^c)_n$ is nested-decreasing, we get that $\mathcal{U}_n^c$ is dense in $\mathcal{S}_{n_0}^d$ for all $n \leq n_0$.

Assume we have shown that $\mathcal{U}_n^c$ is dense in $\mathcal{S}_{n_0}^d$ for some $n \geq n_0$. We now show that $\mathcal{U}_{n+1}^c$ is dense in $\mathcal{S}_{n_0}^d$ as well.

Fix a finite binary string $\zeta \in 2^{<\omega}$ having an infinite extension $Z \in \mathcal{S}_{n_0}^d \cap \mathcal{U}_n^c$. Since the set $\mathcal{U}_n^c$ is open, we may choose an extension $\zeta \preceq \zeta' \prec Z$ for which $[\zeta'] \cap \mathcal{S}_{n_0}^d \subseteq \mathcal{U}_n^c$. Now, as $[\zeta'] \cap \mathcal{S}_{n_0}^d \neq \emptyset$ (Z is contained in their intersection), Lemma 5.3.11 implies there exists an n-thin sequence $Y \in [\zeta'] \cap \mathcal{S}_{n_0}^d \cap \mathcal{T}_n$. And by the choice of ζ', we have $Y \in \mathcal{T}_n \cap \mathcal{U}_n^c$, so by Lemma 5.3.10, $Y \in \mathcal{U}_{n+1}^c$. This shows that $\mathcal{U}_{n+1}^c$ is dense in $\mathcal{S}_{n_0}^d$. The result now follows by induction. $\square$

Chapter 6
Conclusion and Open Questions

Algorithmic information theory continues to offer insights into geometric measure theory. In this dissertation, we have reviewed a subset of the existing results, methods, and notions relevant for witnessing this connection. We have proved some extra robustness and relations in the effective framework, and have extended AIT to a broader class of metric spaces. Namely, net spaces are a productive setting over which to work with algorithmic information theory. And effective notions most closely simulate their classical counterparts over net spaces which are sufficiently rich with net measures. This is exemplified by the various point-to-set principles.

6.1 Dependence on the Net or Dense Subset

Certain notions in algorithmic information theory explicitly depend on the choice of a computable dense subset (or net) with respect to which approximations are taken. For instance, as introduced in Section 1.7, the standard lift of prefix complexity to Euclidean space bases an arbitrary subset's complexity solely on the rational points it contains. Similarly, N. Lutz's locally-optimal outer measure $\boldsymbol{\kappa}(X) := 2^{-K(X)}$ satisfies $\boldsymbol{\kappa}(X) = \boldsymbol{\kappa}(X \cap \mathbb{Q}^m)$, as well as some finiteness properties specific to $\mathbb{Q}^m$. How would another choice of computable dense subset affect these notions of complexity and measure? Lemma 4.8 in [35] makes clear that multiplicative domination (in the sense of Definition 4.4.14) on dyadic cubes is equivalent to multiplicative domination on the collection of open balls of dyadic rational radius; and their Theorem 4.5 implies that their choice of $\boldsymbol{\kappa}$ based on $\mathbb{Q}^m$ dominates on both these nets. But for other choices of computable dense subset, it is not clear that the corresponding lift of prefix complexity would assign comparable complexities to each subset of $\mathbb{R}^m$, nor comparable measures to each subset according to the corresponding outer measure $\boldsymbol{\kappa}$. Thus, the local dimension notion associated to $\boldsymbol{\kappa}$

would not necessarily match with other effective dimension notions. Analogous questions may be asked over generic net spaces.

6.2 Conditional, Mutual Dimension

In Section 1.10, we reviewed two variants of effective dimension: *mutual* and *conditional* dimensions. Notice that for finitary objects, it was indeed possible to define a *conditional, mutual information* of the form $I(a : b \mid c)$: this intuitively measures the information content of the pair of objects a and b when given the data c. But mutual dimension is only defined in the unconditional sense. Can a useful notion of conditional, mutual dimension be defined? If $m, n, \ell \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^\ell$, and $r, s, t \in \omega$, then one reasonable candidate definition would be:

$$\mathrm{mdim}(x : y \mid z) := \liminf_{r \to \infty} \frac{I_{r:r|r}(x : y \mid z)}{r}, \qquad I_{r:s|t}(x : y|z) := \min_{u} \left\{ \min_{p,q} \{I(p : q \mid u)\} \right\},$$

where the choices from p, q, and u range in: $p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m$, $q \in B_{2^{-s}}(y) \cap \mathbb{Q}^n$, and $u \in B_{2^{-t}}(z) \cap \mathbb{Q}^\ell$. The definition for $I_{r:s|t}$ is analogous to that of the conditional prefix complexity $K_{r|s}$.

Recall that conditional complexity admits an approximation by K-minimizers. That is, $K_{r|s}(x \mid y) \approx K(p^* \mid q^*)$, where p^* and q^* are K-minimizers of the balls $B_{2^{-r}}(x)$ and $B_{2^{-s}}(y)$, respectively (see Proposition 2.1.9). But in the case of conditional, mutual information, there is not as clear of a path towards showing the analogous statement: $I_{r:s|t}(x : y \mid z) \overset{?}{\approx} I(p^* : q^* \mid w^*)$, where each p^*, q^*, and w^* are K-minimizers of the appropriate balls about x, y, and z, respectively. To be specific, while a quick calculation shows the approximate inequality $I_{r:s|t}(x : y \mid z) \geq I(p^* : q^* \mid w^*)$ in one direction, we are missing a proof of the reverse direction. The main issue is the lack of known results analogous to the linear sensitivities of conditional complexity: Lemmas 2.1.3 and 2.1.4.

If the approximation to $I_{r:s|t}$ by K-minimizers does hold up to logarithmic terms, then an analogous result to Theorem 2.1.11 should follow for mutual information, and one could go on to define a *conditional, mutual dimension* and compare it to the other variants to effective dimension.

6.3 Applications of CALFs

Recall from Chapter 3 that a continuous absolutely Lipschitz family (CALF) generalizes the uniform continuity properties satisfied by the family of non-vertical planar lines expressed in slope-intercept form. Generalizing the results of N. Lutz and D. Stull for planar lines, we have shown that every CALF admits both a finitary theorem (3.3.3) on the Kolmogorov complexity of points along its graph, as well as an infinitary theorem (3.4.2) on the effective dimension of points along its graph.

In [40], N. Lutz and D. Stull used their infinitary theorem (Theorem 1.1) and the Point-to-Set Principle 1.12.1 to demonstrate a new lower-bound for the Hausdorff dimension of generalized Furstenberg sets. Given $d, \delta \in [0,1]$, the class of generalized Furstenberg sets for d and δ is denoted $F_{d,\delta}$, and a subset $X \subseteq \mathbb{R}^2$ qualifies for $X \in F_{d,\delta}$ if there is some $J \subseteq \mathbb{S}^1$ such that $\dim_{\mathrm{H}} J \geq d$ and such that for any $e \in J$, there exists a line l_e in the direction e for which $\dim_{\mathrm{H}}(X \cap l_e) \geq \delta$. Then given $d, \delta > 0$ and $X \in F_{d,\delta}$, Theorem 4.3 in [40] states $\dim_{\mathrm{H}} X \geq \delta + \min\{d, \delta\}$.

For a more general family of maps such as a CALF Φ on a domain $\Omega \times \Xi \subseteq \mathbb{R}^m \times \mathbb{R}^\ell$, one could similarly define a collection of subsets $F^{\Phi}_{d,\delta}$ for any $0 \leq d \leq m$ and $0 \leq \delta \leq \ell$ as follows: $X \subseteq \mathbb{R}^n$ satisfies $X \in F^{\Phi}_{d,\delta}$ whenever there exists $J \subseteq \Omega$ with $\dim_{\mathrm{H}} J \geq d$ and for all $\boldsymbol{\alpha} \in J$, there exists another $\boldsymbol{\alpha}' \in \Omega$ with $\dim(\boldsymbol{\alpha}') = \dim(\boldsymbol{\alpha})$ such that $\dim_{\mathrm{H}}(X \cap \Phi^{\boldsymbol{\alpha}'}) \geq \delta$. It is not clear for a given CALF Φ whether $F^{\Phi}_{d,\delta}$ will be non-empty, or whether $F^{\Phi}_{d,\delta}$ will have a useful interpretation in the context of geometric measure theory. But the proof method of Theorem 4.3 in [40] likely extends for such sets for non-empty $F^{\Phi}_{d,\delta}$ when $d, \delta > 0$, and should make use of Theorem 3.4.2.

Each new example of a CALF provides a new geometry over which to consider possible implications for classical geometric measure theory. Certain families of planar isometries are natural candidates for CALFs. While the scaling Lipschitz continuity property is more or less clear for families of isometries, we run into difficulties demonstrating the other conditions. For instance, proving the scaling co-Lipschitz continuous differences property for the family of planar rotations is made difficult by the situation that the difference between two angle parameters can be large while the angular difference between them is small. Additionally, it is not clear that one may parameterize the family of planar reflections in a 1-dimensional subspace on an open domain fit for describing them by a CALF.

It is possible that certain results from classical geometric measure theory admit corresponding finitary results. That is, we imagine any statement about the Hausdorff dimension of sets in Euclidean space having a corresponding statement about the Kolmogorov complexity of finitary inputs and outputs as in Theorem 3.3.3. And that finitary result should imply a similar statement but for effective dimension as in Theorem 3.4.2. And by applying the Point-to-Set Principle, one could recover the classical result for Hausdorff dimension. We hope to see more examples of this finitary-infinitary structure for more results in GMT.

6.4 Effectivity for the Subsets of Exact Measure

As discussed in Section 5.3, we have extened Besicovitch's result about finding compact subsets of finite, non-zero s-measure to sufficiently nice nets in our Theorem 5.3.4. The proof of this result relies on a density result: Proposition 5.3.6.

The *n-thin* sequences defined in our proof play the crucial role of helping pass from $\mathcal{U}_n^c$ to $\mathcal{U}_{n_1}^c$, as shown in Lemma 5.3.10. They have the special property that any length-n string used in a cover of E_Z may be replaced by proper extensions to make another cover of E_Z not significantly greater in its direct ρ_h-weight. The n-thin sequences were shown to be dense in any non-empty $\mathcal{S}_{n_0}^d$ whenever $n \geq n_0$ in Lemma 5.3.11. We suspect that the n-thin sequences should be useful in other contexts involving making refinements to optimal covers.

One could also imagine effectivizing the proof of Theorem 5.3.4 as follows: finding an oracle $B \in 2^\omega$ such that, if $F \subseteq 2^\omega$ were in fact a Π_1^0-class, B is strong enough to compute a code $Z \in \mathcal{P}_F$ for which E_Z satisfies the desired properties, putting $E_Z \in \Pi_1^0(B)$. One might even modify $\phi_{n,m}(Z)$ to instead optimize over the *a priori* discrepancy $\Delta \mathrm{M}_s(\sigma)$ over the strings for which $Z(\sigma) = 1$. Then, an appropriate lightface version (i.e., correspondence principle) of Lutz and Miller's point-to-set principle 4.6.7 would relate this function back to the Hausdorff s-measure of the compact subset E_Z coded by Z. In particular, we hope for an effective proof of Besicovitch's result over Cantor space via lower-semicomputable continuous semimeasures.

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Employment

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Honors and Awards

- *Charles H. Hoover Memorial Award.* 2024.
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- *Teaching Associate Status.* 2023.
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- *U.S. Foreign Language and Area Studies Fellowship.* 2023.
- *Robert and Betsy Wilson Endowed Scholarship.* 2021.
- *John & Elizabeth Holmes Teas Scholarship in Science.* 2018.
- *Boyd A. and Bernice K. Mullen Awards in Mathematics.* 2018.
- *Elsbach Honors Scholarship in Physics.* 2017.

Publications

1. *The Algorithmic Theory of Information and its Applications to Geometric Measure Theory.*
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Service

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