CONSTRUCTING ℝEALITY
MAY

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Abstract. There are multiple methods of constructing $\mathbb{R}$, and each way unveils unique mathematics surrounding the process. Thank you to D. E. Knuth for the inspiration.

Contents

1. Introduction

YouTube takes up much of my free time. I pride myself in having exhausted every worthwhile science and math YouTube channel (namely, Veratasium, MinutePhysics, Numberphile, VSauce, Suckerpinch, Vihart, Computerphile, Singingbanana, Carykh, ...). Still, I dislike the sentiment that many of my classmates display when they watch an eight minute video about Electromagnetism and then proclaim themselves experts in the subject. YouTube is powerful for inspirational purposes, but if the video is illustrative and captivating, it is probably not too rigorous. So when Numberphile debuted Donald Knuth and his book *Surreal Numbers*, I wanted to learn more, and was inspired to purchase the book and learn more. That and further readings have since caused me to reevaluate my understanding of real analysis as well, because there seem to be so many constructions or characterizations of $\mathbb{R}$ that are not as far-reaching as that from the surreal framework.

I had a lot of trouble finishing this paper because of my lack of focus. My favorite quote from one of my readings came from Conway himself: "Only several weeks' hard thought, sustained by the conviction that there *must* be a 'genetic' definition, finally led to the 'correct' formula. The genetic definition of $1/x$ at the end of Chapter 1 only appeared a year later."

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2. Surreal Numbers

Donald Knuth created his small booklet about the Surreal Numbers in the style of a dialogue between the wonderfully naïve, yet universally productive across many fields, Alice and Bill. Their story is an organic reaction to some expository rock dictating laws for the order relation $\leq$ and the condition of a *number*. Alice and Bill struggle their way through the theory of surreal numbers: beginning in their lowly skepticism of the process, but then becoming fascinated and anticipating more than they realize. They find more definitions for addition, multiplication, and such, and construct all of $\mathbb{R}$, and more! The style of this book was purposefully designed to teach the method of research to students, as a sort of antithesis to the standard rigorous and linear style of modern textbooks and papers. Knuth uses this start-from-the-bottom method to expose the natural inclinations for many people to choose naïve approaches when first meeting a problem, but then with more practice or experience, returning to old notes and revising to become more rigorous (or not incorrect).

Take for example Alice and Bill's desire to use the argument of day-sums to prove many of their theorems. This was the initial proof to their $T13$.

Theorem 2.1 (T13)**.** *For any given $x, y \in \boldsymbol{No}$ such that $x \leq y$, we have for any $z \in \boldsymbol{No}$,*

$$x + z \leq y + z.$$

Incorrect Proof. This is equivalent to: given $X_L < y$ and $x < Y_R$, we must prove that $X_L + z < y + z, Z_L + x < y + z, x + z < Y_R + z$, and $x + z < Z_R + y$. By day-sum induction, it follows. □

The problem with the above proof is that the induction method on day-sum can only verify for sure the statement $X_L + z \leq y + z$, and so on; it's conceivable that $x_L < y$ but $x_L + z \equiv y + z$. Therefore, we require the converse of $T13$: if $x + z \leq y + z$, then $x \leq y$. Again, the converse is equivalent to being given $X_L + z < y + z, Z_L + x < y + z, x + z < Y_R + z$, and $x + z < Z_R + y$, and having to prove that $X_L < y$ and $x < Y_R$. There may conceivably be a case in the induction process where, say $x_L + z < y + z$ but $x_L \equiv y$. Such cases would be ruled about by $T13$. Thus, we see a codependence between $T13$ and its converse.

Incorrect proof. Take the conjoined statement ($T13$ and $T13'$) can be proved by induction on the day sum of (x, y, z). □

This proof seems legitimate, but it glosses over another case: when breaking the statement $Z_L + x < y + z$ into its two necessary and sufficient cases: $Z_L + x \leq Z_L + y$ and $Z_L + y < z + y$. Induction provides the first step nicely, but the second part involves (z_L, z, y), which could have a larger day sum than (x, y, z).

Next is another attempt at proving $T13$.

Incorrect Proof. Bill arranges six statements to make his notation clear.

I $x + y$ is a number.
II if $x \leq y$, then $x + z \leq y + z$.
III if $x + z \leq y + z$, then $x \leq y$.
IV Combined statement of II and III.
V if $x \leq x'$ and $y \leq y'$, then $x + y \leq x' + y'$.
VI if $x + y \leq x' + y'$ and $y \leq y'$, then $x \geq x'$.

We see items V and VI generalize items II and III, respectively. $I(x,y)$ follows from a list of other inductive statements: $I(X_L,y), I(x,Y_L), I(X_R,y), I(x,Y_R)$, $III(X_R,X_L,y), III(x,X_L,y), II(y,Y_R,x), III(y,Y_L,x), II(x,X_R,y), III(Y_R,Y_L,x)$. We actually see that the rest of the statements are independent of I, so we allow I to be a corollary of the rest. We realize that the combined statement $IV(x,y,z)$ depends on $IV(z,z_L,y)$, which depends on $IV(y_R,y,z)$, which depends on $IV(z,z_L,y)$ again. This loop cannot be broken. However, once we generalize to statements V and VI, we see $V(x,x',y,y')$ depends on $VI(X_L,x',y,y'), VI(Y_L,y',x,x')$, $VI(x,X'_R,y,y'), VI(y,Y'_R,x,x')$. Finally, to prove $VI(x,x',y,y')$, we need, $V(x,X'_L,y,y'), V(X_R,x',y,y')$. Now we can combine the statements with an "and", and by day-sum induction, they hold together! This proof even proves the statements for all *pseudo-numbers*. □

There is beauty to the structure of this proof. However, when day-sums become infinite, this proof does not hold. We can modify our day-sum arguments in the following way: if a theorem fails for some x, then it also fails for some element x_L in X_L, and then also fails in subsequent parts. But if each sequence is eventually finite, as in if eventually we reach a case with $X_{LLRLRR...RLLR}$ being empty, then the theorem can't have failed for x. All we have to do now is show that there is no infinite ancestral sequence of numbers $x_1, x_2, x_3, ...$ such that x_{i+1} is in $X_{iL} \cup X_{iR}$, all with some undesired property. Actually, every pseudo-number is created out of previously created ones. Whenever we create a new number x, we could prove simultaneously that there is no infinite ancestral sequence starting with $x_1 = x$, because we have previously proved that there's no infinite sequence that proceeds from any of the possible choices of x_2 in either X_L or X_R. We notice that rule (1) is the axiom that provides the footing for this logic. And this explanation covers (finite) multi-variable cases because for some (x,y,z), our proofs always involve a permutation of the variables, as a permutation of the variables, where one is given an extra L or R subscript. If there is a chain of infinite permutations, then there has to be an infinite chain for one of the variables, a contradiction to rule (1). We conclude with the realization of a difference between *calculation* and *proof*.

3. Classical Analysis

There are a few methods of constructing $\mathbb{R}$ that are taught to undergraduates especially. I learned a method beginning with the Peano axioms for the natural numbers $\mathbb{Z}^+$.

Axioms 3.1 (Peano). **(N1)** $1 \in \mathbb{Z}^+$,
- **(N2)** if $n \in \mathbb{Z}^+$, then $n+1 \in \mathbb{Z}^+$,
- **(N3)** 1 is not a successor of any element in $\mathbb{Z}^+$, i.e. there exists no $n \in \mathbb{Z}^+$ such that $n+1=1$,
- **(N4)** if $m,n \in \mathbb{Z}^+$ and $m+1 = n+1$, then $m=n$,
- **(N5)** if $S \subset \mathbb{Z}^+$ is such that $1 \in S$ and $n \in S \Rightarrow n+1 \in S$, then $S = \mathbb{Z}^+$.

Next, we construct another set of ordered pairs of elements of $\mathbb{Z}^+$, sorted into equivalence classes based on the relation $(m,n) \sim (s,t)$ iff $m+t = n+s$. We label the set of equivalence classes $\mathbb{Z}$. Next, we construct $\mathbb{Q} = \left\{ \frac{m}{n} : m,n \in \mathbb{Z}, n \neq 0 \right\}$, where $\frac{m}{n}$ represents the equivalence class of the ordered pair (m,n) under the relation $(m,n) \sim (s,t)$ iff $mt = ns$. It is easy to check that this is an equivalence relation,

but the proof of the transitive property requires the use of the cancellation law for integers.

Theorem 3.2 (Cancellation Law)**.** *Suppose that $a, b, c \in \mathbb{Z}$, and $c \neq 0$. Then if $ac = bc$, we have $a = b$.*

Lemma 3.3. *If $a \in \mathbb{Z}$, then*

(i) $a \cdot 0 = 0$*;*
(ii) $-(-a) = a$*.*

Proof. For (i), we use distributivity:

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.$$

Adding the inverse $-(a \cdot 0)$ to both sides gives the result.
For (ii), the statement is equivalently to proving $(-a) + a = 0$, but we have by definition of additive inverse that $a + (-a) = 0$, so commutativity completes the proof. □

Lemma 3.4. *If $a, b, c \in \mathbb{Z}$, then*

(i) $-a = a \cdot (-1)$*;*
(ii) $(-a) \cdot b = -(a \cdot b) = a \cdot (-b)$*;*
(iii) $(a - b) \cdot c = a \cdot c - b \cdot c$*.*

Proof. For (i), we want to show that $a + a \cdot (-1) = 0$, but by the previous lemma (i) and distributivity,

$$a + a \cdot (-1) = a \cdot 1 + a \cdot (-1) = a \cdot (1 + -1) = a \cdot 0 = 0.$$

Now for (ii), we apply (i) repeatedly

$$a \cdot (-b) = a \cdot (b \cdot (-1)) = (a \cdot b) \cdot (-1) = -(a \cdot b).$$

And the other way also follows.
For (iii), we apply part (ii) and distributivity

$$(a - b) \cdot c = (a + (-b)) \cdot c = a \cdot c + (-b) \cdot c = a \cdot c + (-(b \cdot c)) = a \cdot c - b \cdot c.$$

□

Proof of Cancellation Law. We are given $a \cdot c = b \cdot c$, and the previous lemma (iii) gives

$$0 = a \cdot c - b \cdot c = (a - b) \cdot c.$$

The statement $a = b$ is equivalent to $a - b = 0$. By trichotomy, there are three cases to consider: positivity for $a - b$, positivity for $-(a - b)$, or $a - b = 0$. We are given $c \neq 0$, so either c is positive or $-c$ is so. We can check one case and imply the rest by symmetry. Suppose $a - b$ is positive and c is positive. Then by closure $(a - b) \cdot c$ is positive, contradicting that it is equal to 0. If $a - b$ is positive and $-c$ is positive, then by the previous lemma (ii), we have $(a - b) \cdot (-c) = -((a - b) \cdot c)$ is positive, so again it is not zero. The two cases for $-(a - b) > 0$ are handled identically, and we conclude that the only possibility is that $a - b = 0$, which implies $(a - b) \cdot c = 0$. □

This constructive type of approach requires some axioms defining our operations. A synthetic approach to constructing the reals relies on finding the field(s) satisfying the following axioms. We summarize some of the necessary axioms for our operations, and go further than what was necessary, all to introduce $+, \cdot$. They satisfy the field axioms.

Axioms 3.5 (Addition)**.** **(A1)** $x + y = y + x$ for all x, y,
(A2) $x + (y + z) = (x + y) + z$ for all x, y, z,
(A3) there exists an $e = 0$ such that $x + 0 = x$ for all x,
(A4) for all x, there exists x' such that $x + x' = 0$.

Axioms 3.6 (Multiplication)**.** **(M1)** $x \cdot y = y \cdot x$ for all x, y,
(M2) $x \cdot (y \cdot z) = (xy)z$ for all x, y, z,
(M3) there exists an $e = 1 \neq 0$ such that $x \cdot 1 = 1 \cdot x = x$ for all x,
(M4) for all $x \neq 0$, there exists x' such that $x \cdot x' = x' \cdot x = 1$.

Axioms 3.7 (Distributive)**.** **(D1)** $x \cdot (y + z) = x \cdot y + x \cdot z$ for all x, y, z.

Next, we impose the order axioms.

Axioms 3.8 (Order)**.** **(O1)** For any two elements x, y, either $x < y$, $x > y$, or $x = y$,
(O2) If $x < y$ and $y < z$, then $x < z$,
(O3) If $y < z$, then $x + y < x + z$ for all x,
(O4) If $0 < x$, and $0 < y$, then $0 < xy$.

Finally, the distinguishing factor between $\mathbb{R}$ and a subfield such as $\mathbb{Q}$ comes from the Dedekind-completeness axiom.

Axioms 3.9 (Completeness)**.** **(C1)** Every nonempty subset S of the field we are considering that is bounded above has a least upper bound $\sup S$ in the field.

Definition 3.10. A Dedekind cut α is a subset of $\mathbb{Q}$ such that

- $\alpha \neq 0, \mathbb{Q}$;
- If $p \in \alpha$ and $q \in \mathbb{Q}$ satisfies $q < p$, then $q \in \alpha$; and
- If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

We can then define $\mathbb{R} := \{\alpha : \alpha \text{ is a cut}\}$. We order cuts α, β by the rule $\alpha < \beta$ if and only if $\alpha \subset \beta$ strictly. We could check this is truly an order. Addition of cuts occurs by taking

$$\alpha + \beta := \{r + s : r \in \alpha, s \in \beta\}.$$

We could check this form is still a cut, and that addition satisfies the field properties. If both cuts are positive (i.e. $\alpha, \beta > 0$), then their product is

$$\alpha \cdot \beta := \{p \in \mathbb{Q} : p < r \cdot s \text{ for some } r \in \alpha, s \in \beta\}.$$

We define general multiplication in a tedious way.

- $0 \cdot \beta = 0$;
- $\alpha \cdot 0 = 0$;
- If $\alpha < 0, \beta > 0$, $\alpha \cdot \beta = -((-\alpha) \cdot \beta)$;
- If $\alpha > 0, \beta < 0$, $\alpha \cdot \beta = -(\alpha \cdot (-\beta))$;
- If $\alpha < 0, \beta < 0$, $\alpha \cdot \beta = ((-\alpha) \cdot (-\beta))$;

We could check that multiplication satisfies the field axioms as well. Finally, we can define the supremum of a group of cuts as the union: $\sup \alpha_1, \alpha_2, ... = \bigcup \alpha_n$. We end with the conclusion that $\mathbb{R}$ is complete and our process terminates.

4. Cauchy Sequences

Assume we have already constructed $\mathbb{Q}$. The use of Cauchy sequences to construct $\mathbb{R}$ is surprisingly fruitful if we can extend our construction to a more general analysis of all possible absolute value functions on $\mathbb{Q}$. However, let us begin with the usual treatment.

Definition 4.1. A Cauchy sequence of rational numbers is a sequence $(x_1, x_2, ...)$ such that for every $\epsilon \in \mathbb{Q}^+$ there exists $N_\epsilon \in \mathbb{Z}^+$ such that $|x_m - x_n| < \epsilon$ for all $m, n \geq N_\epsilon$.

One may show that every Cauchy sequence is bounded, and that Cauchy sequences may be added and multiplied straightforwardly. Also, all Cauchy sequences have additive inverses, namely $-(x_n) = (-x_n)$. So Cauchy sequences form a commutative ring. However, only Cauchy sequences not equivalent to zero have multiplicative inverses. We define a Cauchy sequence (x_n) as equivalent to zero if and only if $\lim_{n\to\infty} |x_n| = 0$. We can then define two Cauchy sequences $(x_n), (y_n)$ as equivalent if their difference is equivalent to zero. This is an equivalence relation and allows us to add or multiply representatives that preserve convergence. Because of this, the set of equivalence classes of Cauchy sequences forms a field. Finally, we define $\mathbb{R}$ as this set, with $\mathbb{Q}$ embedded via the injection map $[] : x \mapsto [(x, x, x, ...)]$, while extending the absolute value of $\mathbb{Q}$ to $\mathbb{R}$ via $|[(x_n)]| := [(|x_n|)]$. Notice $\mathbb{R}^+ = \{x \in \mathbb{R} \setminus \{0\} : |x| = x\}$, and $\mathbb{R} = -\mathbb{R}^+ \sqcup \{0\} \sqcup \mathbb{R}^+$. Now that we have constructed $\mathbb{R}$ as a completion of $\mathbb{Q}$ with respect to Cauchy sequences from the typical absolute value $|\cdot|$, let's focus on finding *other* completions of $\mathbb{Q}$.

Definition 4.2. An absolute value function on the field k is $||\cdot|| : k \to \mathbb{R}^+$, satisfying

- $||x|| = 0$ iff $x = 0$;
- $||xy|| = ||x||\,||y||$;
- $||x + y|| \leq ||x|| + ||y||$.

A Cauchy sequence can be defined for any absolute value by replacing the condition $|x_m - x_n| < \epsilon$ with $||x_m - x_n|| < \epsilon$.

Definition 4.3. The completion of a field k with respect to absolute value $||\cdot||$ is the field $\widehat{k}$ of equivalence classes of Cauchy sequences of elements of k. We can view k as a subfield of $\widehat{k}$ via the injective map $[] : x \mapsto [(x, x, x, ...)]$ and extend the absolute value on k to $\widehat{k}$ by defining

$$||[(x_n)]|| = [(||x_n||)] = \lim_{n\to\infty} ||x_n|| \in \mathbb{R}^+.$$

Field k with absolute value $||\cdot||$ is called complete if every Cauchy sequence of elements of k converges.

Lemma 4.4. *Field k is dense in its completion $\widehat{k}$.*

Proof. Given $x = [(x_n)] \in \widehat{k}$ and some $\epsilon > 0$, let $r = x_{N_\epsilon}$, where N_ϵ is from the definition of a Cauchy sequence. Then $||x - r|| < \epsilon$, or $(\epsilon - ||x_n - r||)_n \subset \widehat{k}$ is a Cauchy sequence. □

Theorem 4.5. *Completion $\widehat{k}$ of field k with absolute value $||\cdot||$ has the property that each Cauchy sequence (x_n) of $\widehat{k}$ with strictly elements within the image of k under the injective map $[]$ converge to the element $[(x_n)]$ in $\widehat{k}$. Further, $\widehat{k}$ is complete.*

Proof. Let $z = [(r_n)]$ for Cauchy sequence of elements of k, (r_n). Given $\epsilon > 0$, let N_ϵ be as in the definition of Cauchy sequences. Then $||x_n - z|| < \epsilon$ for all $n \geq N_\epsilon$. Now, given a Cauchy sequence (x_n), let (r_n) be a sequence of elements from k such that $||x_n - r_n|| < 1/n$ for all n. Finding such a sequence is possible because k is dense in $\widehat{k}$. Then (r_n) is Cauchy and $\lim_{n\to\infty} ||x_n - r_n|| = 0$. It follows that (x_n) converges to a number in $\widehat{k}$ if and only if (r_n) does, and we know (r_n) converges to a number within $\widehat{k}$ from the first part of the theorem. □

Now let's define a new instance of an absolute value on $\mathbb{Q}$. Let p be a prime number. Then we define the *p-adic absolute value* $|\cdot|_p$ as $|x|_p := p^{-v_p(x)}$, where $v_p(x)$ is the *p-adic valuation* of x. We will define $v_p(x) := e_p$, where e_p comes from the unique representation of any rational number x as a finite product of prime powers:

$$x = \pm \prod_p p^{e_p},$$

as guaranteed by the fundamental theorem of arithmetic.
We also put $v_p(0) := \infty$ and define $p^{-\infty} := 0$ so that $|0|_p = p^{-v_p(0)} = p^{-\infty} = 0$. One can check that $|\cdot|_p$ satisfies the qualities of a non-Archimedean absolute value over $\mathbb{Q}$, thus with the stronger inequality $|x + y|_p \leq \max(|x|_p, |y|_p)$. (The trivial absolute value is non-Archimedean, and the typical absolute value is essentially the only Archimedean absolute value on $\mathbb{Q}$).
To review, we defined $\mathbb{R} = \widehat{\mathbb{Q}}$, the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|$. Similarly, we will denote with $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. We will also define the p-adic integers $\mathbb{Z}_p := \left\{x \in \mathbb{Q}_p : |x|_p \leq 1\right\}$, which forms a subring of $\mathbb{Q}_p$ containing $\mathbb{Z}$ (closure follows from the non-Archimedean property). Also, we see the p-adic absolute values are discrete; i.e., if $r = |x|_p$, and for $\epsilon > 0$ sufficiently small, the real interval $(r - \epsilon, r + \epsilon) \subset \mathbb{R}$ contains no p-adic absolute values other than r. This is clearly true for $x \in \mathbb{Q}$ since $v_p(x) \in \mathbb{Z}$, but is also true for all $x \in \mathbb{Q}_p$ since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$. This allows us to extend the valuation $v_p(x) := -\log_p |x|_p$ for any nonzero $x \in \mathbb{Q}_p$. Thus, $x = p^{v_p(x)}u$ for some unit p-adic number u.

This section ends by using a sequence of statements to show that the only non-trivial completions of $\mathbb{Q}$ are $\mathbb{R}$ and every $\mathbb{Q}_p$. Alling's paper shows how to prove that every p-adic unit can be represented as a Cauchy sequence of integers; specifically, as

$$x = (d_0, d_0 + d_1 p, d_0 + d_1 p + d_2 p^2, ...),$$

where each $d_n \in \{0, ..., p-1\}$ and $d_0 \neq 0$. The converse holds too. Therefore, Alling decides to use the notation $0.d_0 d_1 d_2 ..._p$ to notate each p-adic unit, and explains all other p-adic numbers are just powers of p off of some unit, justifying a shift in the "decimal" point. This representation allows for the application of diagonalization, proving each $\mathbb{Q}_p$ is uncountable. One interesting example of how to represent a p-adic number is -1 in $\mathbb{Q}_p$. Actually, $-1 = 0.\overline{(p-1)}_p$, and this can be justified

using the formula for a geometric series:

$$\begin{aligned} x &= (p-1) + (p-1)p + (p-1)p^2 + ... \\ &= (p-1)\left(\frac{1}{1-p}\right) \\ &= -1. \end{aligned}$$

Another good example is trying to calculate the p-adic representation of $\sqrt{-1}$, say, in $\mathbb{Q}_5$. Alling explains how to deduce the first digit simply by assuming $\sqrt{-1} = 0.d_0d_1d_2...{}_5 : (d_0 + d_1 5 + ...)^2 = \sqrt{-1}^2 = 0.44\overline{4}$. We see $d_0 = 2$. Then we find d_1:

$$0.\overline{4} = (2 + d_1 5 + ...)^2 = 4 + 4d_1 5 + ...,$$

so $d_1 = 1$. Continuing, we find $\sqrt{-1} = 0.212134230322041324....$ The fact that $\sqrt{-1} \in \mathbb{Q}_5$ shows that $\mathbb{Q}_5$ is not isomorphic to $\mathbb{R}$. Alling also describes the conditions for finding r^{th} roots of some integer $z \in \mathbb{Q}_p$, which I will leave out. Finally, Alling provides proof of the following theorem.

Theorem 4.6. *The nontrivial completions of $\mathbb{Q}$ are $\mathbb{R}$ and the p-adic fields $\mathbb{Q}_p$.*

I could also provide the proof, but I believe the purpose of this paper is to study *how* an argument is made, or how a paper is written, etc. So I will provide you with my synopsis of the proof. Alling first makes clear some universal truths about an arbitrary absolute value on $\mathbb{Q}$. Namely, $||\pm 1|| = 1$, and for some distinct p, q primes,

$$||p||^n \leq (m+1)q \max(||q||, ||q||^m),$$

where $m = \lceil \log_q p^n \rceil$. Alling then breaks the proof into two cases; we see $\max(||q||, ||q||^m)$ is present because $||p||$ may be less than or more than 1. The first case considers the existence of at least one prime p such that $||p|| > 1$. It follows that $||q|| > 1$ for all other prime q. We then use our inequality to establish an inequality version of the line below, but it works in both directions by symmetry of the argument, converting it to the equality:

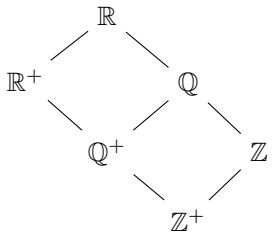
$$\frac{\log ||p||}{\log p} = \frac{\log ||q||}{\log q}.$$

Thus, let $\alpha = \frac{\log||p||}{\log p}$, meaning for any prime, $||q|| = |q|^\alpha$; then $||x|| = |x|^\alpha$ for any $x \in \mathbb{Q}_p \setminus \{0\}$. Alling states that a sequence of rational numbers is Cauchy with respect to the absolute value $||\cdot|| = |\cdot|^\alpha$ if and only if it is also Cauchy with respect to $|\cdot|$, meaning the completion of $\mathbb{Q}$ in this case is isomorphic to $\mathbb{R}$. In the second case, we suppose no prime p has the property $||p|| > 1$. If all primes have absolute value 1, then we have the trivial absolute value with the trivial completion. Otherwise, there exists some prime p such that $||p|| < 1$. Alling shows that if this case, all other primes have absolute value 1. He does this by assuming the existence of another prime with the same property $||q|| < 1$, and by making use of some number theoretic facts, concluding with a contradiction that $1 < 1$. Thus, because of the multiplicativity of absolute value, $||x|| = ||p||^{v_p(x)}$, only depending on p, for any nonzero $x \in \mathbb{Q}$. Writing $||p|| = p^{-\alpha}$ for some $\alpha > 0$ gives $||x|| = |a|_p^\alpha$. Thus, we get exactly the same equivalence classes of Cauchy sequences using $||\cdot|| = |\cdot|_p^\alpha$ as we do with $|\cdot|_p$, meaning the completion of $\mathbb{Q}$ is with respect to $||\cdot||$ is $\mathbb{Q}_p$, completing the proof.

Alling makes some concluding remarks: While the fields $\mathbb{R}$ and $\mathbb{Q}_p$ are complete topologically, they are still incomplete algebraically, as they are missing solutions to some polynomial equations. Taking the algebraic closure of $\mathbb{R}$ is as easy as adding the element $i = \sqrt{-1}$; together with $\mathbb{R}$ this yields the field $\mathbb{C}$, which is still complete with respect to the Archimedean absolute value. However, things are not as simple for p-adic spaces, because after taking the algebraic closure of $\mathbb{Q}_p$, we yield a space $\overline{\mathbb{Q}_p}$ which is no longer complete. Taking another Cauchy completion of this spaces yields $\mathbb{C}_p$, which is still algebraically closed. Like $\mathbb{C}$, $\mathbb{C}_p$ is the smallest extension of $\mathbb{Q}$ that is both closed and complete with respect to the extension of an absolute value on $\mathbb{Q}$.

5. The Logical Theory of Real Numbers

I found this diagram in Conway's *On Numbers and Games*, where it was used to describe the methods of constructing $\mathbb{R}$.



If we suppose $\mathbb{Z}^+$ already constructed, then we have $\binom{3}{2} = 3$ possible paths towards constructing $\mathbb{R}$. Conway claims the best path is to follow $\mathbb{Z}^+ \to \mathbb{Q}^+ \to \mathbb{R}^+ \to \mathbb{R}$. The process of constructing $\mathbb{X} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ from $\mathbb{X}^+$ involves the introduction of ordered pairs (a, b) (which we may interpret as $a - b$) and the equivalence relation $(a, b) \sim (c, d)$ iff $a + d = b + c$. There is an alternative approach to adding 0 and $-x$ but that requires case work, which we aim to avoid.

Similarly, we can proceed from $\mathbb{Z}$ to $\mathbb{Q}$ or $\mathbb{Z}^+$ to $\mathbb{Q}^+$ by introducing ordered pairs (a, b) (which we may interpret as a/b) and the equivalence relation $(a, b) \sim (c, d)$ iff $ad = bc$. The usual methods of proceeding from $\mathbb{Q}$ to $\mathbb{R}$ or their positives include the use of Dedekind cuts or Cauchy sequences. Dedekind cuts require at least 4 special cases in the definition of the product xy based on the signs of $x, y \in \mathbb{Q}$. Moreover, 0 requires its own statements. Thus, we would have 8 cases in proof of the associative law $(xy)z = x(yz)$ and strictly more in the distributive law $(x + y)z = xz + yz$.

Because of this, case handling using Dedekind cuts seems to arise when sign is introduced, so we intend to prolong the introduction until the end, thus justifying following the unique path through $\mathbb{R}^+ \to \mathbb{R}$. Conway expresses the opinion that Cauchy sequences are much too heavy of weaponry for simply constructing $\mathbb{R}$ out of $\mathbb{Q}$. He believes sequences belong to real analysis, which only begins after $\mathbb{R}$ is constructed!

However, there are disadvantages to considering the surreal approach. One, ironically, is the relative fruitfulness of the process. The process never seems to stop after constructing $\mathbb{R}$. Conway shows in a paper of his that the surreal numbers form the Field **No**, a proper class and real, closed field, with a very high level of

density, which can be descried by extending Hausdorff's n_ξ condition. **No** is totally-ordered and non-Archimedean, meaning the relation of $\leq$ is partial: reflexive, anti-symmetrical, and transitive; for all x, y, either $x \leq y$ or $y \leq x$; and we do not have the Archimedean property that given any $g, h > 0$, there exists some $n \in \mathbb{Z}^+$ such that $ng > h$ and $nh > g$. Conway suggests the cure to this is adding to the construction a stipulation: if L is non-empty but with no greatest member, then R is non-empty with no least member, and vice versa. This condition gives us a unique real number $x = \sup L = \inf R$, and nothing greater than some integer, for example. There are some further disadvantages, one being the special treatment of the dyadic rationals, and the inductive definitions of $1/x, \sqrt{x}, ...$ He explains that this prevents him from teaching undergraduates this as "the" theory of real numbers, but he does provide one last suggestion.

Conway's surreal numbers define addition and multiplication in the follow ways.

$$x + y = \left\{x^L + y, x + y^L : x^R + y, x + y^R\right\},$$

$$\begin{aligned} xy = \{&x^L y + xy^L - x^L y^L,\ x^R y + xy^R - x^R y^R \mid \\ &\mid x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}. \end{aligned}$$

On day ω, all non-dyadic real numbers are born. For example,

$$\frac{1}{4} < \frac{1}{4} + \frac{1}{16} < \frac{1}{4} + \frac{1}{16} + \frac{1}{64} < ... < \frac{1}{3} < ... < \frac{1}{2} - \frac{1}{8} < \frac{1}{2},$$

so we might expect that $\left\{\frac{1}{4}, \frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{1}{16} + \frac{1}{64}, ... : \frac{1}{2}, \frac{1}{2} - \frac{1}{8}, ...\right\} = x = \frac{1}{3}$. In fact, one can check that $x + x + x = 1$. Similarly, all of the real numbers defined by Dedekind including all of the remaining rational numbers may be defined as Dedekind sections or cuts of the dyadic rational numbers, rather than as sections of *all* rationals. Now for any dyadic rational such as $\frac{3}{8}$ may be represented as

$$\left\{\text{dyadic rationals } < \frac{3}{8} : \text{dyadic rationals } > \frac{3}{8}\right\},$$

and $\frac{3}{8} = \left\{\frac{1}{4} : \frac{1}{2}\right\}$, meaning all dyadic rationals recreated on day ω are the equal to those created in preceding days.

Conway suggests beginning with a classical approach up to and including $\mathbb{Q}$. All rational numbers, when viewed in surreal form, are sections of dyadic rationals. We then define the reals as sections of $\mathbb{Q}$ with the definitions of addition and multiplication provided, and then all the formal laws have "1-line" proofs and there is no case-splitting.

6. Rambling Echos

I used the opportunity of writing this paper to more purposefully analyze how an author organizes content, especially in how an argument is structure to prove a statement. Alling's paper especially helped in this front, and I had similar training in preparing my Algebra final for the MASS program last semester on Quaternion Algebras.

I've had a lot of professors already in my two years "at university." Each one enjoys spouting tidbits of wisdom on how to best present information. My professor for topology, Prof. Anton Petrunin, is someone from whom I could not only hear wisdom in the classroom, but someone I could find online. MathOverflow allows me to

see his account, his activity, his comments, his posts, his influence on the entire community. He seems to primarily promote ideas of free dissemination of information between colleagues. It is astounding how some middle-state, average, agricultural university could host some of the many most important mathematicians. Professor Petrunin would frequently drive a point about writing mathematics: sometimes authors become fixated on being perfectly rigorous and detailed in their writing; but the point of writing in mathematics is to say enough so that the reader understands, and then nothing more!

The majority of my professors favor accurate, organized, and terse writing for technical pieces. This is far from that, but is meant to prepare me for it. A post-doctoral student, John Pretz, was the first person I found at Penn State to help me with my writing. The thing he said that best stuck with me read along the lines of "if you can read this sentence the same way as before after taking out this one word, then take it out." That extinguished my liking towards "certainly" or "however."

Professor Svetlana Katok favored all of the aspects I already mentioned, but also valued *content*. Paragraphs filled with bluff or little new information were superfluous in her mind. It is a view with which I agree, although my idea of new information might be more liberal compared to her's.

Knuth wanted to highlight the process of creating content in his book, and I think having reread this book twice has allowed me to appreciate the process, especially after having also read so many technical papers in the past year. Textbooks attempt to derive and motivate their topics, but they still lack the discovery and thought-intensive troubleshooting sessions. All I do is troubleshoot, really. And shoot, I have got plenty of troubles.

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