DISCREPANCY THEORY
MARCH

RAYMOND FRIEND

Abstract. Yes, this math is very abstract.

Contents

1. Introduction

This paper was inspired by a video by Terence Tao. Much of these notes were compiled from Karen R. Johannson from University of Manitoba. This subject was something I never realized had so much attention, and it seems like some of the best mathematicians flock to this subject. I want to do more with this in the future.

2. Erdős Discrepancy Problem

One notoriously difficult problem that required a very unique and groundbreaking approach to solve by Terence Tao in 2016 is the Erdős Discrepancy Problem, analyzing boundedness on homogeneous sums in the range $\{-1, 1\}$.

Problem 2.1. *Given infinite sign sequence $(f(j))_{j=1}^{\infty} \subset \{-1, 1\}$, and any $d \in \mathbb{N}$, are the following sums always unbounded?*

$$\sum_{j=1}^{\infty} f(jd).$$

Equivalently, given any $C \geq 0$, we ask if there exists $N > 0$ such that for every sign sequence $(f(j))_{j=1}^{N} \subset \{-1, 1\}, \exists d, k \in \mathbb{N}$ such that

$$\left| \sum_{j=1}^{k} f(jd) \right| \geq C.$$

Now I will begin an exposition into solving for the corresponding N for small values of C.

Date: March 14: π Day, 2017.

Definition 2.2. Sign sequence $f(1), f(2), ..., f(n) \in \{-1, 1\}$, or discrete function $f : [1, n] \to \{-1, 1\}$ is *well distributed* with respect to $C \in \mathbb{N}$ if and only if $\forall d \in \{1, ..., n\}$ and $k \in \left[1, \left\lfloor \frac{n}{d} \right\rfloor\right]$,

$$\left| \sum_{j=1}^{k} f(jd) \right| < C.$$

Definition 2.3. Integer $n \in \mathbb{N}$ is *well distributive* with respect to $C \in \mathbb{N}$ if and only if $\exists f : [1, n] \to \{-1, 1\}$ that is well distributed with respect to C.

Lemma 2.4. *If $\exists N \in \mathbb{N}$ that is not well distributive with respect to fixed $C \in \mathbb{N}$, then $\forall n > N$, n is not well distributive with respect to C.*

Proof. Suppose $n > N$ is well distributive with respect to C, i.e. $\exists f' : [1, n] \to \{-1, 1\}$ that is well distributed with respect to C. Then $\forall d' \in [1, n] \supset [1, N]$ and k',

$$\left| \sum_{j=1}^{k'} f'(jd') \right| < C.$$

However, $f := f'|_{[1,N]}$ is thereby well distributed with respect to C as $\forall d \in [1, N]$ and k,

$$\left| \sum_{j=1}^{k} f(jd) \right| = \left| \sum_{j=1}^{k'} f'|_{[1,N]} (jd') \right| < C,$$

for corresponding $d' = d, k' = k$. Obviously, we have a contradiction. $\square$

Lemma 2.5. *If $\exists N \in \mathbb{N}$ that is well distributive with respect to fixed $C \in \mathbb{N}$, then $\forall n \in [1, N]$, n is well distributive with respect to C.*

Proof. We have N is well distributive with respect to C, so $\exists f : [1, N] \to \{-1, 1\}$ that is well distributed with respect to C. Then $\forall d \in \{1, ..., N\}$ and k,

$$\left| \sum_{j=1}^{k} f(jd) \right| < C.$$

If $n < N$, we notice that $f' := f|_{[1,n]}$ is also well distributed with respect to C, since $\forall d' \in [1, n] \subset [1, N]$ and k', we have corresponding $d = d'$ and $k = k'$ such that

$$\left| \sum_{j=1}^{k'} f'(jd') \right| = \left| \sum_{j=1}^{k} f|_{[1,n]} (jd) \right| < C.$$

$\square$

From the last two lemmas, we know that for any mapping $f : \mathbb{N} \to \mathbb{Z}$, f may only be well distributed with respect to any $C \in \mathbb{N}$ over the integers $1, 2, ..., N$, where N could be finite or ∞ in principle. The Erdős Problem asks whether N is always finite. There are a few known results in this vein. First, if $C = 1$, any sign sequence of length $n = 1$ guarantees the sum having magnitude equal to C, so $N = 0$. Next, we will see $N = 11$ is the largest well distributive number with respect to $C = 2$.

Example 2.6. *Prove $N = 12$ is not well distributive with respect to $C = 2$.*

Proof. Suppose $\exists f : [1, 12] \to \{-1, 1\}$ such that f is well distributed with respect to 2. Then let us assume $f(1) = a \in \{-1, 1\}$. As a bit of notation, we will consider inequalities including sums of the form

$$\left| \sum_{j=1}^{k} f(jd) \right| < 2,$$

and shorten the statement of this inequality to (k, d). First, we notice $(2, 1)$ implies $f(2) = -a$, or else $|f(1) + f(2)| = |2a| = 2$, which is not less than $C = 2$. We continue this chain of arguments in a Sudoku-like fashion:

$$\begin{aligned}
(2,2) \Rightarrow & f(4) = -f(2) = a \\
(2,4) \Rightarrow & f(8) = -f(4) = -a \\
(4,2) \Rightarrow & f(6) = -f(8) = a \\
(2,3) \Rightarrow & f(3) = -f(6) = -a \\
(6,1) \Rightarrow & f(5) = -f(6) = -a \\
(8,1) \Rightarrow & f(7) = -f(8) = a \\
(2,5) \Rightarrow & f(10) = -f(5) = a \\
(1,10) \Rightarrow & f(9) = -f(10) = -a \\
(2,6) \Rightarrow & f(12) = -f(6) = -a.
\end{aligned}$$

However, $(4, 3) \Rightarrow f(12) = -f(9) = a$. This is a contradiction, and thus 12 is not well distributive with respect to $C = 2$. $\square$

Corollary 2.7. *A natural observation from this proof is that we can construct $f : [1, 11] \to \{-1, 1\}$ that is well distributed with respect to $C = 2$. Notice we showed in the previous proof that such an f must satisfy*

$$\begin{aligned}
a = & f(1) = f(4) = f(6) = f(7) = f(10), \\
-a = & f(2) = f(3) = f(5) = f(8) = f(9).
\end{aligned}$$

Because 11 *is prime and over half of* 12*, only* $(k = 11, d = 1)$ *can imply the value of* $f(11)$*. But notice* $f(11) = \pm 1$ *works since*

$$\left| \sum_{j=1}^{11} f(j) \right| = |0 + f(11)| = |\pm 1| = 1 < 2.$$

Thus, we have four possible functions over $[1, 11]$ *that are well distributed with respect to* $C = 2$*, corresponding to choices from* $a = \pm 1$ *and* $f(11) = \pm 1$*.*

According to Alexei Lisitsa and Boris Konev of the University of Liverpool, $N = 1160$ is the maximal integer that is well distributive with respect to $C = 3$. For $C = 4$, the maximal well distributive integer is definitely greater than $130,000$.

3. Naïve Proofs

One cute and naïve result in infinite sums relates to the misunderstood extension of the Riemann-Zeta Function. First, we begin with the function $f : \mathbb{N} \to \{-1, 1\}$, such that $f(n) = (-1)^{n-1}$. We take the sum

$$\sum_{n \in \mathbb{N}} f(n) = \sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + ...$$

Let's denote the value of this sum as $S_1 = 1 - 1 + 1 - 1 + ...$ Then let's add S_1 to itself in a shifted manner:

$$\begin{aligned} S_1 &= 1 - 1 + 1 - 1 + 1 - 1 + ... \\ +S_1 &= 0 + 1 - 1 + 1 - 1 + 1 - ... \\ \Rightarrow 2S_1 &= 1 + 0 + 0 + 0 + ... = 1. \end{aligned}$$

We achieve $S_1 = \frac{1}{2}$. Furthermore, we can use this to imply more. For instance, if we denote $S_2 = 1 - 2 + 3 - 4 + 5 - 6 + ...$, then we can solve for the value of S_2 performing a similar trick

$$\begin{aligned} S_2 &= 0 + 1 - 2 + 3 - 4 + 5 - 6 + ... \\ +S_2 &= 1 - 2 + 3 - 4 + 5 - .... \\ \Rightarrow 2S_2 &= 1 - 1 + 1 - 1 + ... = S_1. \end{aligned}$$

Thus, $S_2 = \frac{1}{4}$. Finally, if we denote $S_3 = 1+2+3+4+5+6+...$, then we perform

$$\begin{aligned} S_3 &= 1 + 2 + 3 + 4 + 5 + 6 + ... \\ -S_2 &= -1 + 2 - 3 + 4 - 5 + 6 - .... \\ \Rightarrow S_3 + \frac{1}{4} &= 0 + 4 + 0 + 8 + ... = 4S_3. \end{aligned}$$

Thus, $S_3 = 1+2+3+4+5+... = -\frac{1}{12}$. Obviously, this connects to the most basic definition of Riemann's ζ function, but is an invalid process to obtain the result since we assumed each sum had a determined final value.

4. Ramsey's Theorem

Definition 4.1. An *arithmetic progression* of length $k \geq 3$ is a sequence of k numbers of the form $a, a+d, a+2d, ..., a+(k-1)d$, and denoted AP_k.

We think of grouping the elements of a set by a *partition* and into *partition classes*. We generally call Ramsey theory the study of structures preserved under partition. A density result known as Szemerédi's theorem states that for any $k \geq 3$ and $\epsilon \in (0, 1)$, $\exists n \in \mathbb{N}$ such that $S \subset [1, n]^{\epsilon n}$ (subset of size ϵn) will contain a k-term arithmetic progression. For any set A and finite set B, any function $\Delta : A \to B$ is called a finite coloring or finite partition of A. There are $\left|\left\{\Delta^{-1}(b) \mid b \in B\right\}\right|$ many partition/color classes. The *pigeonhole principle* says that if $n, r \in \mathbb{N}$, then for every r-coloring $\Delta : [1, nr+1] \to [1, r]$, there exists $i \in [1, r]$ so that $\left|\Delta^{-1}(i)\right| \geq n+1$.

Theorem 4.2 (Ramsey, 1930)**.** *For every $k, m, r \in \mathbb{N}$ with $k \leq m$, there exists integer $n \in \mathbb{N}$ so that for any set S with $|S| = n$ and any r-coloring $\Delta : [S]^k \to [1, r]$ there is a set $T \in [S]^m$ such that $[T]^k$ is monochromatic; i.e. $\exists i \in [1, r]$ such that $i = \Delta(t)$ for all $t \in [T]^k$.*

Raysey's Theorem states if there is a finite coloring from the collection of k-subsets of some S, then there is some set T from the collection of larger subsets of S such that the coloring applied to each k-subset of T produces the same color.

Definition 4.3. For non-empty V and $E \subset V^2$, the pair $G = (V, E)$ is called a *graph*. Elements of V are *vertices* and thsoe of E are *edges*. The *neighborhood* of $v \in V$ is $N(v) = \{x \in V \mid \{x, v\} \in E\}$.

We call G complete if $E = V^2$. We will denote the complete graph of n vertices with K_n.

Definition 4.4. Given graph $G = (V, E)$, any graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E \cap [V']^2$ is called a *subgraph* of G.

We can explore the $m = 2$ case of Ramsey's theorem using graph theory.

Theorem 4.5. *For every $k, r \in \mathbb{N}, \exists R(k; r)$ such that for all $n \geq R(k; r)$ and for any r-coloring of the edges of K_n, there is a complete subgraph G on k vertices such that $E(G)$ is monochromatic.*

Definition 4.6. Let $r, k_1, k_2, ..., k_r \in \mathbb{N}$, and denote $R(k_1, k_2, ..., k_r)$ the least integer N, if it exists, such that for every $n \geq N$ and any r-coloring of the edges of K_n, there is an $i \in [1, r]$ such that K_n contains a subgraph K_{k_i} whose edges are monochromatic in color i.

If every $R(k_1, k_2, ..., k_r)$ exists, then the Ramsey number $R(k; r) = R(k, k, ...k; r)$ exists as well. We claim R for r positive integer arguments is invariant up to permutation.

Lemma 4.7. *Let $k_1, ..., k_r \in \mathbb{N}$ be such that $R(k1, ..., k_r)$ exists. For any permutation $\sigma \in S_r$, $R\left(k_{\sigma(1)}, ..., k_{\sigma(r)}\right) = R(k_1, ..., k_r)$.*

Proof. Set $N = R(k_1, ..., k_r)$, and let Δ be any r-coloring of $E(K_N)$. Define a new r-coloring Δ' of the edges of K_N as follows. For $\{x, y\} \in E(K_N)$, set $\Delta'(\{x, y\}) = \sigma(\Delta(\{x, y\}))$. By the choice of N, for some $i \in [1, r]$, there is K_k that is monochromatic under Δ' in the color i; i.e. $\forall x, y \in V(K_{k_i}), \sigma(\Delta(\{x, y\})) = i$. Let $j = \sigma^{-1}(i)$. Then $K_{k_i} = K_{k_{\sigma(j)}}$ is a complete graph under Δ of color j. $\square$

A form of recursion known as Erdős-Szekeres recursion was employed to prove Ramsey's theorem and discussed finding convex k-gons from any collection of N line-free sets.

Theorem 4.8 (Erdős-Szekeres Recursion)**.** *For all integers $r \geq 2$ and $k_1, k_2, ..., k_r \geq 3$,*

$$\begin{aligned} R(k_1, k_2, ..., k_r) \leq & R(k_1 - 1, k_2, ..., k_r) + R(k_1, k_2 - 1, ..., k_r) \\ & + ... + R(k_1, k_2, ..., k_r - 1) - r + 2. \end{aligned}$$

Proof. For each $i \in [1, r]$, define $N_i = R(k_1, ..., k_i - 1, ..., k_r)$ and let $N = N_1 + ... + N_r - r + 2$. We claim $R(k_1, ..., k_r) \leq N$. Let Δ be any r-coloring of the edges K_N. Fix $x \in V(K_N)$ and for each i, define

$$V_i = \{y \in V(K_N) \mid \Delta(\{x, y\}) = i\}.$$

Then $V_i \subset N(x)$ is all edges in neighborhood of x colored i by Δ. Then $\exists \ell \in [1, r]$ such that $|V_i| \geq N_i$. Otherwise,

$$\left(\sum_{i=1}^{r} N_i\right) - r + 2 = 1 + \sum_{i=1}^{r} |V_i| \leq 1 + \sum_{i=1}^{r} (N_i - 1) = \left(\sum_{i=1}^{r} N_i\right) - r + 1,$$

a contradiction. Fix ℓ such that $|V_\ell| \geq N_\ell$. The coloring Δ induces an r-coloring of the complete graph on the vertices of V_ℓ and so by the choice of the number $N_\ell = R(k1, ..., k_\ell - 1, ..., k_r)$, either for some $j \in [1, r] \setminus \{\ell\}$, V_ℓ contains a K_{k_j} with edges all color j, or else V_ℓ contains a $K_{k_\ell - 1}$ with edges all color ℓ. In the latter, the vertices of the complete $K_{k_\ell - 1}$ together with the vertex x form the vertices of a complete graph on k_ℓ vertices all of whose edges are color ℓ since all of the edges between x and V_ℓ are of color ℓ by construction of V_ℓ. □

Corollary 4.9. *Every $R(k_1, ..., k_r)$ exists for all $r, k_1, ..., k_r \in \mathbb{N}$.*

Proof. We check the necessary base cases for double induction on r and $k_1 + ... + k_r$. For $r = 2$ and $k \in \mathbb{N}$, $R(k, 2) = k$ since for any 2-coloring of the edges of the complete graph K_k, there is either one edge (K_2) of one color or else all edges of the graph K_k are of the other color. Similarly, if $k_1, ..., k_{r-1} \in \mathbb{N}$, then $R(k_1, ..., k_{r-1}, 2) = R(k_1, ..., k_{r-1})$ (Check for yourself). The previous lemma and theorem then both imply each $R(k_1, ..., k_r)$ exists. □

Remark 4.10. *Actually, we can place the numerical bound when $k, m \geq 2$: $R(k, m) \leq \binom{k+m-2}{k-1}$, and it follows from the recursion of the previous theorem. Base case $R(k, 2) = k = \binom{k+2-2}{k-1}$. Then by the Erdős-Szekeres Recursion theorem,*

$$\begin{aligned}
R(k, m) \leq R(k-1, m) + R(k, m-1) - 2 + 2 &\leq \binom{k+m-3}{k-2} + \binom{k+m-3}{k-1} \\
= (k+m-3)! \left[\frac{1}{(k-2)!(m-1)!} \frac{k-1}{k-1} + \frac{1}{(k-1)!(m-2)!} \frac{m-1}{m-1} \right] \\
= \frac{(k+m-3)!}{(k-1)!(m-1)!} [k+m-2] = \binom{k+m-2}{k-1}.
\end{aligned}$$

5. Schur's Theorem

Theorem 5.1. *For every $r \in \mathbb{N}$ there is a least positive integer $S(r)$ such that for any r-coloring $\Delta : [1, S(r)] \to [1, r]$, there exist $x, y \in [1, S(r)]$, possibly with $x = y$, such that $\Delta(x) = \Delta(y) = \Delta(x + y)$.*

Proof. Let $r \in \mathbb{N}$ and set $n = R(3; r) - 1$. To see that $S(r) \leq n$, let $\Delta : [1, n] \to [1, n]$ be any r-coloring and consider the graph K_{n+1} on vertices $0, 1, ..., n$ with an edge coloring defined as follows. For $0 \leq i < j \leq n$,

$$\Delta^*(\{i, j\}) = \Delta(j - i).$$

With this choice of i and j, $j - i \in [1, n]$ and thus Δ^* is well-defined. By the choice of n, there is a triangle in K_{n+1} which is monochromatic under Δ^*. Thus, in terms of Δ, there are $0 \leq a < b < c \leq n$ such that

$$\Delta(b - a) = \Delta(c - b) = \Delta(c - a) = \Delta((b - a) + (c - b)).$$

If $x = b - a, y = c - b, z = c - a$, we see $\Delta(x) = \Delta(y) = \Delta(z = x + y)$. □

This proof places a bound on $S(r) \leq R(3; r) - 1$, but the original proof of this theorem, which did not use Ramsey's theorem, found the better bound $S(r) \leq er!$.

6. Van der Waerden's Theorem

Theorem 6.1 (Van der Waerden)**.** *For all $N \in \mathbb{N}$ and $f : \mathbb{N} \to [1, N]$ and for any $k \in \mathbb{Z}$, $\exists i \in [1, N]$ and integers $a, d > 0$ such that*

$$\{a, a + d, a + 2d, ..., a + (k-1)d\} \subset f^{-1}(i).$$

For all possible ways of splitting the natural numbers into N groups, at least one group will contain an arithmetic progression of any specified length k.

Equivalently,

Theorem 6.2 (Van der Waerden, Alt.)**.** *For every $k, r \in \mathbb{N}$, there is an integer n such that for every r-coloring of $[1, n]$, there is a monochromatic AP_k.*

Remark 6.3. *There is no similar result for infinite arithmetic progressions. One example that expresses this notion is the 2-coloring of $\mathbb{N}$ defined by $\Delta(n) = \lfloor \log_2 n \rfloor \pmod 2$. For this coloring, the interval $\left[2^i, 2^{i+1} - 1\right]$ is of color 0 if i is even, and of color 1 otherwise. Both classes contain arbitrarily long arithmetic progressions with finitely many terms, but no infinite arithmetic progressions.*

Lemma 6.4. *For any $n \in \mathbb{N}$ and $k > 1$, the number of AP_k in $[1, n]$ is less than $\frac{n^2}{2(k-1)}$.*

Proof. For any integer $d \in \left[1, \frac{n-1}{k-1}\right]$, the arithmetic progressions in $[1, n]$ with difference d are $\{1, 1+d, ..., (k-1)d\}$, ..., $\{n - (k-1)d, ..., n\}$. So the number of AP_k with difference d is $n - (k-1)d$. Therefore, the total number of AP_k contained in $[1, n]$ is

$$\begin{aligned}
\sum_{d=1}^{\lfloor \frac{n-1}{k-1} \rfloor} (n - (k-1)d) &= \left\lfloor \frac{n-1}{k-1} \right\rfloor n - (k-1) \binom{\left\lfloor \frac{n-1}{k-1} \right\rfloor + 1}{2} \\
&= \left\lfloor \frac{n-1}{k-1} \right\rfloor \frac{1}{2} \left(2n - (k-1) \left(\left\lfloor \frac{n-1}{k-1} \right\rfloor + 1 \right) \right) \\
&\leq \frac{n-1}{2(k-1)} (2n - (n-1)) \\
&= \frac{(n-1)(n+1)}{2(k-1)} \\
&< \frac{n^2}{2(k-1)}.
\end{aligned}$$

□

While working at Hamburg in 1926, Bartel van der Waerden apparently shared a conjecture from Baudet in Göttingen on arithmetic progressions with Artin and Shrier. It was likely due to Schur originally, and stated that for every $k \geq 3$, any partition of $\mathbb{N}$ into **two** classes will contain an AP_k. We claim Theorem 6.2 is equivalent to the original conjecture. But we need some preliminaries to prove this.

Definition 6.5. For every $k, r \in \mathbb{N}$ let $W(k; r)$ be the least integer, if it exists, so that for every r-coloring of $[1, W(k; r)]$, there are $a, d \in \mathbb{N}$ so that the arithmetic progression of length k starting at a is monochromatic. The numbers $W(k; r)$ are called *van der Waerden numbers*.

Lemma 6.6 (Schrier)**.** *Fix $r, k \in \mathbb{N}$. The integer $W(k;r)$ exists if and only if for every r-coloring of $\mathbb{N}$, there is a monochromatic AP_k.*

Proof. Obviously, if $W(k;r)$ exists, then for every r-coloring of $\mathbb{N}$ there is a monochromatic AP_k since $[1, W(k;r)] \subseteq \mathbb{N}$.
Conversely, we will prove the contrapositive. Suppose $W(k;r)$ does not exist. Then for every $n \in \mathbb{N}$, there is an r-coloring Δ_n for which $[1,n]$ contains no monochromatic AP_k. These r-colorings are used to construct an r-coloring of $\mathbb{N}$ with no monochromatic AP_k. Recursively build a sequence of colors $\{c_n\}_{n\in\mathbb{N}}$ and a sequence of infinite sets $A_1 \supseteq A_2 \supseteq ...$ as follows.
Since there are only finitely many colors, one color must occur infinitely many times in the sequence $\{\Delta_1(1), \Delta_2(1), ...\}$. Let $c_1 \in [1,r]$ be such that $A_1 = \{i \in \mathbb{N} \mid \Delta_i(1) = c_1\}$ is infinite. In general, for $t \geq 1$, having defined the infinite set A_t, there must be one color, c_{t+1}, that occurs infinitely many times in the sequence $\{\Delta_i(t+1) \mid i \in A_t\}$.
Now we define a new coloring $\Delta : \mathbb{N} \to [1,r]$ by $\Delta(n) = c_n$. Note that for $m, n \in \mathbb{N}$, if $n \in A_m$, then $\Delta_n|_{[1,m]} = \Delta|_{[1,m]}$ by the definition of Δ and the choice of the set A_m. For each $n \in \mathbb{N}$, there are no AP_k which are monochromatic under Δ_n and so $\mathbb{N}$ also does not contain any AP_k which are monochromatic under Δ. $\square$

Lemma 6.7. *Fix $k, r \in \mathbb{N}$, and suppose that the number $n = W(k;r)$ exists and let $P = \{a, a+d, ..., a+(n-1)d\}$ be any AP_n. Then for any r-coloring of P, there is a monochromatic $AP_k \subset AP_n$.*

Proof. Let $\Delta : P \to [1,r]$ by any r-coloring. Define an induced r-coloring $\Delta^* : [1,n] \to [1,r]$ by $\Delta^*(i) = \Delta(a + d(i-1))$. By the choice of n, there is an AP_k: $\{c, c+b, ..., c+(k-1)b\}$ that is monochromatic under Δ^*. In terms of Δ, for each $0 \leq i \leq k-1$, $\Delta^*(c+ib) = \Delta(a + d(c+ib-1))$. Therefore, the AP_k: $\{a+d(c-1), (a+d(c-1))+db, ..., (a+d(c-1))+(k-1)db\}$ is monochromatic under Δ. $\square$

Lemma 6.8 (Blocks)**.** *Suppose that for some $k, r, n \in \mathbb{N}$, that $W(k;r^n) = N$ exists. For any r-coloring $\Delta : [1, nN] \to [1,r]$, there exists an AP_k of blocks, each block of length n, all with the same color pattern under Δ, were an AP_k of blocks is a sequence $B_1, ..., B_k$ such that $\exists d > 0$ so that for each $i \in [2,k]$, $B_i = B_1 + (i-1)d$.*

Proof. For each $i \in [1,N]$, set $B_i = [1+(i-1)n, in]$. Let $\Delta : [1, nN] \to [1,r]$ be any r-coloring and define the induced r^n-coloring $\Delta^* : [1,N] \to ([1,r])^n$ by

$$\Delta^*(x) = (\Delta(1+(x-1)n), \Delta(2+(x-1)n), ..., \Delta(xn))$$

where each color under Δ^* is an n-tuple. Since $N = W(k;r^n)$, $[1,N]$ contains AP_k: $\{a, a+d, ..., a+(k-1)d\}$ which is monochromatic under Δ^*. In terms of Δ, this means that the blocks $B_a, B_{a+d}, ..., B_{a+(k-1)d}$ form an AP_k of blocks of length n, with difference nd, for which all members have the same color pattern under Δ. $\square$

Lemma 6.9 (Artin)**.** *If for all $k \in \mathbb{N}$, $W(k;2)$ exists, then for all $k \in \mathbb{N}, r \geq 2$, the van der Waerden number $W(k;r)$ exists.*

This lemma reduces the existence of van er Waerden numbers to the existence of van der Waerden numbers for 2-colorings.

Proof. We prove this by induction on r. The base case $r = 2$ is true by assumption. Fix $k \in \mathbb{N}$ and suppose that $r \geq 3$ is such that $W(k;r-1)$ exists. Set $m =$

$W(k;r-1)$ and $n = W(m;2)$. In order to show that $W(k;r) \leq n$, let $\Delta : [1,n] \to \{0_1, 0_2, ..., 0_{r-1}, 1\}$ be any r-coloring. Define an induced 2-coloring $\Delta^* : [1,n] \to [0,1]$ by

$$\Delta^*(i) = \begin{cases} 0 & \text{if for some } j, \Delta(i) = 0_j; \\ 1 & \text{if } \Delta(i) = 1. \end{cases}$$

If $[1,n]$ contains an AP_m under Δ^* of color 1, then since the AP_m will also be of color 1 under $\Delta, [1,n]$ contains a monochromatic AP_k since $k \leq m$.
Otherwise, by the choice of n, there is an AP_m, denoted P, which is of color 0 under Δ^*. Thus, Δ restricted to P is an $(r-1)$-coloring and since $m = W(k;r-1)$, by the previous lemma, P contains an AP_k which is monochromatic under Δ. □

Theorem 6.10 (Brown and Rabung)**.** *Let $M \in \mathbb{N}$ be such that for every $(M-1)$-coloring of $\mathbb{N}$, at least one color class contains arbitrarily long arithmetic progressions. Let $S = \{s_i\})i \geq 0$ be strictly increasing sequence such that for all $i \geq 0$, $|s_{i+1} - s_i| \leq M$. Then S contains arbitrarily long arithmetic progressions.*

Proof. Define a partition of $\mathbb{N}$ into M disjoint sets as follows. Set $A_0 = S$ and for each $n \in [1, M-1]$, set

$$A_n = \{s_i + n \mid i \geq 0\} \setminus \left(\bigcup_{j=0}^{n-1} A_j \right).$$

Since for all $i \geq 0$, $|s_{i+1} - s_i| \leq M$, the sets $A_0, ..., A_{M-1}$ define a partition of $\mathbb{N}$. By assumption, there is one $n \in [1, M-1]$ so that A_{n_0} contains arbitrarily long arithmetic progressions, or for each $k \in \mathbb{N}$, there are $a, d \in \mathbb{N}$ so that $\{a, a+d, ..., a+(k-1)d\} \subseteq A_{n_0} \subseteq S + n_0$. Therefore, the k-term arithmetic progression $\{a - n_0, a - n_0 + d, ..., a - n_0 + (k-1)d\}$ is contained in S. □

Corollary 6.11 (Rabung)**.** *If, for every finite coloring of $\mathbb{N}$, one color class contains arbitrarily long arithmetic progressions, then for any partition of $\mathbb{N}$ into 2 classes, either one class contains arbitrarily long strings of consecutive numbers or else both classes contain arbitrarily long arithmetic progressions.*

Proof. Let $\mathbb{N} = A_1 \cup A_2$ be any partition. If for some $M \in \mathbb{N}$, the longest string of consecutive integers in A_1 is of length M, then for any two consecutive entries $a < b$ in A_2, $|b - a| \leq M + 1$. Therefore, if neither A_1 nor A_2 contain arbitrarily long strings of consecutive numbers, then both A_1 and A_2 satisfy the conditions of Theorem 6.10 and hence contain arbitrarily long arithmetic progressions. □

Finally, we can perform the proof that the van der Waerden numbers exist.

Theorem 6.12. *For each $k, r \geq 2$, $W(k;r)$ exists.*

Proof. We perform induction on k with a recursive construction in r steps. Fix $r \geq 2$. By the pigeonhole principle, $W(2;r) = r+1$. Now we fix $k \geq 2$ and suppose that for all $t \geq 2$, the number $W(k;t)$ exists. The following inductive step shows that $W(k+1;r)$ exists. Set $q_0 = 1$ and for each $s \in [1, r]$, define

$$n_{s-1} = W(k; r^{q_s - 1} \textit{ and } q_s = 2ns - 1q_{s-1}.$$

The goal of the proof is to show that $W(k+1;r) \leq q_r$. Fix an r-coloring $\Delta : [1, q_r] \to [1, r]$. using the choices of n_s and q_s, a sequence of arithmetic progressions of blocks are defined recursively in r steps as follows. Since $q_r = 2n_{r-1}q_{r-1}$, the

interval $[1, q_r]$ can be divided into $2n_{r-1}$ blocks each of length q_{r-1} and since $n_{r-1} = W(k; r^{q_r-1}$, by Lemma , among the first n_{r-1} blocks in $[1, q_r]$, there is an AP_k of blocks: $\{B(1), B(2), ..., B(k)\}$, all with the same color patter under Δ. Set $B(k+1) = B(k) + d_1$.
The blocks $B(1), B(2), ..., B(k)$ are all contained in the first half of the interval $[1, q_r]$, so $B(k+1) \subseteq [1, q_r]$, but nothing is known about the coloring of $B(k+1)$ yet. Since $q_{r-1} = 2n_{r-2}q_{r-2}$ the block $B(1)$ can be divided into $2n_{r-2}$ blocks of length q_{r-2}. Since $n_{r-2} = W(k; r^{q_r-2}$, in the first half of $B(1)$ there is an AP_k of blocks $\{B(1,1), B(1,2), ..., B(1,k)\}$ with difference d_2 which all have the same color pattern under Δ. Set $B(1, k+1) = B(1,k) + d_2$. Since $B(1,k)$ is contained in the first half of the block $B(1), B(1, k+1) \subseteq B(1)$. But again, nothing is known about the coloring of $B(1, k+1)$.
Translate the AP_{k+1} of blocks $\{B(1,1), ..., B(1,k), B(1,k+1)\}$ into the other blocks $B(2), ..., B(k+1)$ as follows: for $i \in [2, k+1]$ and $j \in [1, k+1]$, define

$$B(i,j) = B(1,j) + (i-1)d_1.$$

Since the AP_k of blocks $\{B(1,1), ..., B(1,k)\} \subseteq B(1)$ all have the same color pattern and the blocks $B(1), ..., B(k)$ all have the same color pattern for $1 \leq i \leq j \leq k$ all the blocks $B(i,j)$ have the same color pattern. Also, for $1 \leq i \leq k$, all of the blocks $B(i, k+1)$ have the same color pattern, though not necessarily the same as $B(1,1)$. In general, for $s < r$ at step s of the recursion, the block $B(1, ..., 1)$ will be an interval of length $q_{r-s+1} = 2n_{r-s}q_{r-s}$ and if $B(1, ..., 1)$ is partitioned into $2n_{r-s}$ blocks, since $q_{r-s} = W(k; r^{q_{r-s}}$, the first half of $B(1, ..., 1)$ contains an AP_k of blocks

$$\{B(1, ..., 1, 1), B(1, ..., 1, 2), ..., B(1, ..., 1, k)\}$$

with difference d_s, and all with the same color pattern under Δ. Set

$$B(1, ..., 1, k+1) = B(1, ..., k) + d_s$$

and translate the AP_{k+1} of blocks $\{B(1, ..., 1, 1), ..., B(1, ..., 1, k+1)\}$ into all the blocks constructed in step $s-1$ of the recursion. Note that if $i_1, ..., i_s,\ j_1, ..., j_s \in [1, k]$, then the blocks $B(i_1, i_2, ..., i_s)$ and $B(j_1, ..., j_s)$ have the same color pattern under Δ.
After step r of the recursion, the blocks are all of size $q_0 = 1$. Since these blocks are all singletons, they will be treated interchangeably as integers or sets. The following properties of the blocks of integers constructed in this way are worth noting. First, if $1 \leq s < r$ and if $1 \leq i_{s+1}, i_{s+2}, ..., i_r \leq k+1$, then $B(i_1, ..., i_s, i_{s+1}, ..., i_r)$ appears in the same position in the block $B(i_1, ..., i_s)$ as $B(j_1, ..., j_s, i_{s_1}, ..., i_r)$ does in the block $B(j_1, ..., j_s)$. If $1 \leq i_1, ..., i_s, j_1, ..., j_s \leq k$, then since the two blocks have the same color pattern, the two integers have the same color under Δ.
Also, for $1 \leq s \leq r$, since $B(i_1, ..., i_{s_1}, i_s + 1) = B(i_1, ..., i_{s_1}, i_s) + d_s$ and
$B(i_1, ..., i_{s_1}, i_s, i_{s+1}, ..., i_r)$ and $B(i_1, ..., i_{s_1}, i_s + 1, i_{s+1}, ..., i_r)$ appear in the same position in their respective blocks,

$$B(i_1, ..., i_{s_1}, i_s + 1, i_{s+1}, ..., i_r) - B(i_1, ..., i_s, ..., i_r) = d_s.$$

Consider the following $r+1$ elements. For each $i \in [0, r]$, let

$$b_i = B(\underbrace{1, ..., 1}_{\text{i}}, \underbrace{k+1, ..., k+1}_{\text{r-i}}).$$

Since Δ is an r-coloring, by the pigeonhole principle, there must be u and v with $0 \leq u < v \leq r$ so that $\Delta(b_u) = \Delta(b_v)$. For each $i \in [1, k+1]$, define

$$a_i = B(\underbrace{1, ..., 1}_{\text{u}}, \underbrace{i, ..., i}_{\text{v-u}}, \underbrace{k+1, ..., k+1}_{\text{r-v}}).$$

Then $a_1 = b_v$, and $a_{k+1} = b_u$. If $i + 1 \leq k$, then by a previous remark, $a_i = B(1, ..., 1, i, ..., i, k+1, ..., k+1)$ and $a_i = B(1, ..., 1, i+1, ..., i+1, k+1, ..., k+1)$ have the same color. Since $\Delta(a_1) = \Delta(a_{k+1})$, the set $\{a_1, ..., a_{k+1}\}$ is monochromatic under Δ. To show that $\{a_1, ..., a_{k+1}\}$ is an AP_{k+1}, fix $i \in [1, k]$ and for each $m \in [0, v-u]$, define

$$a_{i_m} = B(\underbrace{1, ..., 1}_{\text{u}}, \underbrace{i+1, ..., i+1}_{\text{m}}, \underbrace{i, ..., i}_{\text{v-u-m}}, \underbrace{k+1, ..., k+1}_{\text{r-v}})$$

so that $a_{i,0} = a_i$, and $a_{i,v-u} = a_{i+1}$. Now

$$\begin{aligned} a_{i,m} - a_{i,m-1} &= B(1, ..., 1, i+1, ..., i+1, \underbrace{i+1}_{\text{(u+m)-th}}, i, ..., i, k+1, ..., k+1) \\ &\quad - B(1, ..., 1, i+1, ..., i+1, \underbrace{i}_{\text{(u+m)-th}}, i, ..., i, k+1, ..., k+1 \\ &= d_{u+m}. \end{aligned}$$

The sequence $\{a_{i,m} \mid 0 \leq m \leq v-u\}$ can be used to write the difference $a_{i+1} - a_i$ as a telescoping series.

$$\begin{aligned} a_{i+1} - a_i &= \sum_{m=1}^{v-u} a_{i,m} - a_{i,m-1} \\ &= \sum_{m=1}^{v-u} d_{u+m} \\ &= d_{u+1} + d_{u+2} + ... + d_v. \end{aligned}$$

Therefore, the set $\{a_1, ..., a_{k+1}\}$ is a monochromatic AP_{k+1} with difference $d_{u+1} + d_{u+2} + ... + d_v$ and so $W(k+1; r) \leq q_r$. Therefore, by induction, for any $k \in N$, the van der Waerden number $W(k; r)$ exists. $\square$

This implies Theorem 6.1.