MÖBIUS TRANSFORMATION TUESDAY JULY

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Abstract. Algebra beautifully encapsulates all of the intricate phenomena of geometry, in what I still find surprising. Geometric intuition is something everyone shares, but algebraic passion is something only a bored child would find himself exuding in the middle of the summer.

Contents

1. Introduction

This paper began with reading *Geometries*; but I am only so confident with the subject because of the exposure I received from both Dr. Svetlana Katok in her book *Fuchsian Groups* and Dr. Serge Tabachnikov's MATH 313H course. Before that course, I graduated high school believing I had at least *heard* of the majority of problems, branches, or terms in mathematics. Dr. Tabachnikov showed me what I had learned was on par with the curriculum of his middle school, and that mathematics was immensely richer than I had expected. Ever since, the internet has contributed to that feeling as well. Playing the Wikipedia game taught me how to navigate the web simply by use of hyperlinks, but the skill has exposed me to pages upon pages of mathematics that I have yet to understand or even hear about. This paper is nothing more than another conglomeration of a few Wikipedia articles and some of their sources. Hopefully a permissible rate of a few Wikipedia pages per month (alongside school) is enough to carry me through these years devoted towards learning that have been assigned to me. On this subject, though, I enjoyed the algebraic quality to this month's work. It has exposed to me the advantages and mental disadvantages of having many equivalent formulations or structures of some basic algebraic objects, such as Möbius transformations.

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2. I've Got My Own Problems, Algebra

A classical theorem of Joseph Liouville determines that the set of all conformal mappings of Euclidean space of degree greater than 2 under the usual metric are compositions of these types of transformations: translations, homotheties, rotations, and inversions. We define such a composition as a Möbius transformation, which must take the form

$$f(x) = b + \frac{\alpha A(x-\alpha)}{|x-a|^{\epsilon}},$$

where $a, b \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, ϵ is either 0 or 2, and A is an orthogonal matrix. Really, the domain of f is the one-point compactification of n-dimensional Euclidean space: $\overline{\mathbb{R}^n}$. Alternatively, we may think about equivalent spaces $\overline{\mathbb{R}^n} \cong \widehat{\mathbb{C}} \cong \mathrm{CP}^1$, the Riemann sphere and complex projective line. Liouville's theorem does not apply to the planar case, but we may extend the idea of a Möbius transformation to this dimension and investigate its properties. Namely, we will define a Möbius transformation on $\overline{\mathbb{R}^2}$ as a function of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. One may show that every Möbius transformation on $\overline{\mathbb{R}^2}$ is in fact a conformal mapping by strenuous verification that the inversion transformation $z \mapsto 1/z$ is conformal, since translations and rotations trivially preserve angles. Moreover, we can imply other qualities of these transformations, including the preservation of the cross-ratio between four points, and the fact that generalized-circles are sent to generalized-circles. The Möbius transformations are the orientation-preserving, bijective, conformal maps from the Riemann sphere to itself, i.e., the orientation-preserving automorphisms of the Riemann sphere as a complex manifold. Therefore, the set of all Möbius transformations forms a group under composition, called the Möbius group, and may be denoted by $\text{Möb}(\mathbb{C}) \cong \text{Aut}^+(\overline{\mathbb{C}})$.

Let us begin by establishing many of the algebraic statements that aid in the classification and complete description of these transformations. First notice the way we have defined Möbius transformations on $\overline{\mathbb{R}^2}$ does in fact produce the complete set of compositions of orientation-preserving isometries, homotheties, and inversions. To see this, notice each type of transformation is Möbius, and then apply each type to a general Möbius transformation to obtain another Möbius transformation. Also, if we are given the transformation $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, then we may calculate its inverse and derivative. The approach for inverse is more clear if we define

$$f_1(z) = z + \frac{d}{c} \qquad f_2(z) = \frac{1}{z}$$

$$f_3(z) = \frac{bc-ad}{c^2} z \qquad f_4(z) = z + \frac{a}{c}.$$

Then since $f = f_4 \circ f_3 \circ f_2 \circ f_1$, we have

$$\begin{aligned} f^{-1}(z) &= f_1^{-1} \circ f_2^{-1} \circ f_3^{-1} \circ f_4^{-1}(z) \\ &= \frac{dz - b}{-cz + a}. \\ f'(z) &= \frac{ad - bc}{(cz + d)^2}. \end{aligned}$$

Note that the condition $ad - bc \neq 0$ is useful because it is exactly the condition for the form $\frac{az+b}{cz+d}$ to equal a constant C. Plugging in $z = 0$, we have $C = \frac{b}{d}$, but $z = 1$ implies $C = \frac{a+b}{c+d}$. Combining, we achieve $ad = bc$. Constant functions are not considered part of our set of Möbius transformations, so we exclude this case. Next, let us consider the fixed points of f. We try to solve for a fixed $z = \gamma$:

$$\begin{aligned} \gamma = f(\gamma) \Leftrightarrow \gamma &= \frac{a\gamma + b}{c\gamma + d} \\ &\Leftrightarrow c\gamma^2 + (d - a)\gamma - b = 0 \\ &\Leftrightarrow \gamma_{1,2} = \frac{(a - d) \pm \sqrt{(a - d)^2 + 4bc}}{2c} \\ &= \frac{(a - d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2c}. \end{aligned}$$

When $c = 0$, we see $\gamma_1 = -\frac{b}{a-d}$, while the other fixed point is at infinity; if $a = d$, then both fixed points are at infinity. Otherwise, we will naturally introduce a classification of Möbius transformations based on the above discriminant. First note that we could identify every element $f \in \text{Möb}(\mathbb{C})$ with an element $\mathfrak{H} \in M(2, \mathbb{C})$ by

$$f(z) = \frac{az + b}{cz + d} \leftrightarrow \mathfrak{H} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

However, we see that $\text{Möb}(\mathbb{C}) \cong \text{GL}(2, \mathbb{C})/\{(\mathbb{C}\setminus\{0\})I\} = \text{PGL}(2, \mathbb{C})$ since a Möbius transformation determines its matrix only up to scalar multiples, and $ad - bc \neq 0$ implies inverability. Similarly, $\text{Möb}(\mathbb{C}) \cong \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm I\}$. In our classification scheme, we can assume $\mathfrak{H}$ to be normalized such that its determinant $\det \mathfrak{H} = ad - bc = 1$. Thus, the expression for the fixed points $\gamma_{1,2}$ reduces to

$$\gamma_{1,2} = \frac{(a - d) \pm \sqrt{(a + d)^2 - 4}}{2c}.$$

Call a Möbius transformation **parabolic** if $\text{tr}^2 \mathfrak{H} = (a + d)^2 = 4$. Besides the identity mapping, which fixes all of $\overline{\mathbb{C}}$, all (and only) parabolic transformations have exactly one fixed point:

$$\gamma = \frac{a - d}{2c}.$$

All non-parabolic transformations must have exactly two fixed points over $\overline{\mathbb{C}}$. An **elliptic** transform is one whose matrix $\mathfrak{H}$ has real trace, satisfying $0 \leq \text{tr}^2 \mathfrak{H} < 4$. The particular case of $\text{tr} \mathfrak{H} = 0$ is denoted as the **circular** transform, and corresponds to a rotation by π about two fixed points. The two fixed points of an elliptic transform are

$$\gamma_{1,2} = \frac{(a - d) \pm i\sqrt{4 - (a + d)^2}}{2c},$$

with those of the circular transform being

$$\gamma_{1,2} = \frac{(a-d) \pm 2i}{2c}.$$

Finally, a **loxodromic** transform corresponds to $\operatorname{tr}^2 \mathfrak{H} \in \mathbb{C} \setminus [0,4]$, with the special case of being **hyperbolic** when its trace is real and $\operatorname{tr}^2 \mathfrak{H} > 4$.

3. Normal Formal

We can express all Möbius transformations in their **normal form**, as functions of their fixed points. Without use of homogeneous coordinates, we require cases dealing with fixed points at infinity. Because the action of $\mathrm{PGL}(2,\mathbb{C})$ on CP^1 is the action of the Möbius group on the Riemann sphere, we identify

$$[z_1 : z_2] \leftrightarrow z_1/z_2.$$

The brackets are homogeneous coordinates (satisfying division relations) on CP^1, where $[1:0]$ corresponds to the point at ∞ on the Riemann sphere. Recall in the non-parabolic case, transform f has exactly 2 (distinct) fixed points in $\overline{\mathbb{C}}$: γ_1, γ_2, where if either 0 or ∞ are fixed points of f, we choose $\gamma_1 = 0$ and/or $\gamma_2 = \infty$. We claim every non-parabolic transformation is conjugate to a dilation or rotation: $z \mapsto kz$ for some $k \in \mathbb{C}$, which has fixed points 0 and ∞. To see this, define the map

$$g(z) = [[z - \gamma_1 : \gamma_2] : [z : \gamma_2] - 1].$$

This map sends γ_1 to 0 and γ_2 to ∞ (in every case of $\phi_{1,2}$). The composition gfg^{-1} thus fixes 0 and ∞, and is a dilation. Thus, the fixed point equation for the transformation f can be written

$$[f(z) - \gamma_1 : f(z) - \gamma_2] = k[z - \gamma_1 : z - \gamma_2].$$

Solving for f gives us

$$f(z) = \left[\left(k - \frac{\gamma_1}{\gamma_2}\right) z + (1-k)\gamma_1 : \frac{k-1}{\gamma_2} z + 1 - \frac{k\gamma_1}{\gamma_2}\right],$$

$$\mathfrak{H}(k; \gamma_1, \gamma_2) = \begin{pmatrix} k - \frac{\gamma_1}{\gamma_2} & (1-k)\gamma_1 \\ \frac{k-1}{\gamma_2} & 1 - \frac{k\gamma_1}{\gamma_2} \end{pmatrix}.$$

The advantage of using homogeneous coordinates is revealed when we consider the case of $\gamma_2 = \infty$; the expression for $\mathfrak{H}$ reduces to

$$\mathfrak{H}(k; \gamma, \infty) = \begin{pmatrix} k & (1-k)\gamma \\ 0 & 1 \end{pmatrix}.$$

One may use these formulas to calculate the derivatives of f at the fixed points, finding

$$f'(\gamma_1) = k, \text{ and} \qquad f'(\gamma_2) = 1/k.$$

We are then well to call k a characteristic constant to f, and reversing the order of the fixed points is equivalent to taking the inverse multiplier for the characteristic constant:

$$\mathfrak{H}(k; \gamma_1, \gamma_2) = \mathfrak{H}(1/k; \gamma_2, \gamma_1).$$

We already showed that all non-parabolic transformations can be defined by a matrix conjugate to

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where the complex number λ is not equal to $0, \pm 1$. This corresponds to a rotation/homethety from multiplication by the number $k = \lambda^2$. Elliptic transformations have multipliers $|k| = 1, k = e^{\pm i\theta} \neq 1$; circular have $k = -1$; hyperbolic have multipliers $k \in \mathbb{R}^+, k = e^{\pm\theta} \neq 1$; and loxodromic generally have multipliers $|k| \neq 1, k = \lambda^2 = \lambda^{-2}$.
In the parabolic case, I have had great difficulty attempting to produce homogeneous representations of suitable g and f that work for any value of the only fixed point $\gamma \in \overline{\mathbb{C}}$. The problem is unique to this case because γ could be 0 or ∞, unlike in the non-parabolic case where we could let γ_1 take 0 and γ_2 take ∞. This is disheartening, but I am forced to attack by means of two sub-cases. We pick the transformation sending our fixed point γ to ∞ as

$$g(z) = \begin{cases} \frac{1}{z-\gamma} & \gamma \neq \infty, \\ z & \gamma = \infty. \end{cases}$$

The composition gfg^{-1} fixes infinity and is therefore a translation: $gfg^{-1}(z) = z+\beta$ for some $\beta \in \mathbb{C}$. When $\gamma \neq \infty$, we see

$$\frac{1}{f(z) - \gamma} = \frac{1}{z - \gamma} + \beta,$$

so we have that

$$\mathfrak{H}(\beta; \gamma) = \begin{pmatrix} 1 + \gamma\beta & -\beta\gamma^2 \\ \beta & 1 - \gamma\beta \end{pmatrix}.$$

If $\gamma = \infty$, $f(z) = z + \beta$ obviously, so

$$\mathfrak{H}(\beta; \infty) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

At the fixed point, the derivative $f'(\gamma) = 1$. Notice the characteristic constant for any parabolic transformation is $k = 1$. Ideally, one could either confirm or edit the choices of g, consolidating them into one choice, and then show some expression for $\mathfrak{H}$ is well-defined (avoids indeterminate expressions) for every case of γ. Now, we may write the characteristic constant of a Möbius transformation in terms of its logarithm: $k = e^{\rho+\alpha i}$. We interpret ρ as an expansion factor, explaining how attractive the fixed point λ_1 is, and how repulsive λ_2 is. The case $\rho = 0$ is precisely elliptic, with zero attraction or repulsion. Points about λ_1, λ_2 are rotated about them. If $\alpha = 0$, this is precisely hyperbolic, with vector fields appearing identical to those of electric field lines between positive and negative electrical charges. Loxodromic transformations require both $\rho = 0$ and $\alpha = 0$, and appear as S-shapes, spiraling about both fixed points.

4. Peaceful Poleitics

I read the entire Wikipedia page on Poland just to research that pun. Let us define the two poles of any non-identity Möbius transformation: $z_\infty = -d/c, Z_\infty = a/c$. The former is the point that f transforms to ∞, and the latter is the point to which f transforms ∞. We see that $z_\infty + Z_\infty = \frac{a-d}{c} = \gamma_1 + \gamma_2$ (where the parabolic case has a root of multiplicity 2). If we were to specify γ_1, γ_2, and z_∞, we use our relations to obtain

$$\mathfrak{H}(z_\infty; \gamma_1, \gamma_2) = \begin{pmatrix} \gamma_1 + \gamma_2 - z_\infty & -\gamma_1\gamma_2 \\ 1 & -z_\infty \end{pmatrix}.$$

Comparing to the previous expression for $\mathfrak{H}$, we are left with more relations:

$$z_\infty = \frac{k\gamma_1 - \gamma_2}{1-k}$$
$$k = \frac{\gamma_2 - z_\infty}{\gamma_1 - z_\infty} = \frac{Z_\infty - \gamma_1}{Z_\infty - \gamma_2} = \frac{a - c\gamma_1}{a - c\gamma_2},$$

which reduces to

$$k = \frac{(a+d) + \sqrt{(a-d)^2 + 4bc}}{(a+d) - \sqrt{(a-d)^2 + 4bc}}.$$

We can also relate the fixed points to the eigenvalues of the matrix representing our transformation: $\mathfrak{H}$. Its characteristic polynomial

$$\begin{aligned}\chi(\lambda) &= \det(\lambda I - \mathfrak{H}) \\ &= \lambda^2 - \operatorname{tr}\mathfrak{H}\cdot\lambda + \det\mathfrak{H} \\ &= \lambda^2 - (a+d)\lambda + (ad-bc)\end{aligned}$$

has roots

$$\lambda_i = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2} = \frac{(a+d) \pm \sqrt{(a+d)^2 + 4(ad-bc)}}{2} = c\gamma_i + d.$$

5. Public Properties

Möbius transformations can be shown to be conformal maps by noting that circle inversion is also a conformal operation. Moreover, generalized circles are sent to generalized circles by the same reasoning. One can also prove that the cross-ratio between four points in the plane is invariant under Möbius transformations; i.e. if four distinct points z_1, z_2, z_3, z_4 are sent to distinct points w_1, w_2, w_3, w_4 by Möbius f, then

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_2 - w_3)(w_1 - w_4)}.$$

Note that if one of the points, say, $z_4 = \infty$, the cross-ratio is modified to be

$$\frac{z_1 - z_3}{z_2 - z_3}.$$

One verifies this fact by testing whether a translation, rotation, and inversion all preserve cross-ratio. These are all straightforward calculations, and immediately imply the same for $\text{Möb}(\mathbb{C})$.

Corollary 5.1. *The cross-ratio of four points is real if and only if they all lie on some generalized circle.*

Proof. We can think of the cross-ratio of four points as a function of the first argument with the other three arguments fixed:

$$T : z \mapsto \frac{(z - z_3)(z_2 - z_4)}{(z_2 - z_3)(z - z_4)}.$$

Notice $T(z_2) = 1, T(z_3) = 0$, and $T(z_4) = \infty$. Since T maps generalized circles to generalized circles and is bijective, T must map the circle or straight line passing through z_2, z_3, z_4 to the the real line with infinity, $\overline{\mathbb{R}}$, and $T(z) \in \overline{\mathbb{R}}$ if and only if $z = T^{-1}Tz$ lies on the circle or straight line passing through z_2, z_3, z_4, which is the image of the real line plus infinity under the Möbius transformation T^{-1}. □

Next, we would like to make a claim on determining a Möbius transformation from a few pre-images and their images.

Theorem 5.2. *The action of the Möbius group on the Riemann sphere is sharply 3-transitive; i.e. given a set of three distinct points z_1, z_2, z_3 on the Riemann sphere and a second set of distinct points w_1, w_2, w_3, there exists exactly one Möbius transformation $f(z)$ with $f(z_i) = w_i$ for each $i = 1, 2, 3$.*

Proof. Let T_1 be a Möbius transformation such that

$$T_1 : z \mapsto \frac{(z - z_2)(z_1 - z_3)}{(z_1 - z_2)(z - z_3)}.$$

Really, we make use of the homogeneous coordinates again to help with preventing casework. Let $\mathfrak{G}_1$ be the matrix of T_1. If one of the points is ∞, then let that be $z_3 = \infty$. We have

$$\mathfrak{G} = \begin{pmatrix} [z_1 - z_3 : z_3] & -z_2[z_1 - z_3 : z_3] \\ [z_1 - z_2 : z_3] & z_2 - z_3 \end{pmatrix}.$$

We have $T_1(z_1) = 1, T_1(z_2) = 0$, and $T_1(z_3) = \infty$. Next, define a similar $\mathfrak{G}_2$ for the map T_2 taking w_1, w_2, w_3 to $0, 1, \infty$, respectively. Then $\mathfrak{H} = \mathfrak{G}_2^{-1}\mathfrak{G}_1$ is the matrix representation of the map taking z_1, z_2, z_3 to w_1, w_2, w_3. □

Alternate proof. This proof gives an explicit formula by means of determinants. If we require that some $w = \frac{az+b}{cz+d}$, this is equivalent to the equation of a standard hyperbola

$$cwz - az + dw - b = 0.$$

Our problem is thereby equivalent to finding the coefficients a, b, c, d of the hyperbola passing through the points (z_i, w_i). An explicit equation can be found by evaluating the determinant

$$\begin{vmatrix} zw & z & w & 1 \\ z_1w_1 & z_1 & w_1 & 1 \\ z_2w_2 & z_2 & w_2 & 1 \\ z_3w_3 & z_3 & w_3 & 1 \end{vmatrix}.$$

Laplace expansion is the method used for this evaluation, and each of the 4 steps corresponds to the result for $c, -a, d$, and $-b$, in that order. For example,

$$c = \begin{vmatrix} z1 & w_1 & 1 \\ z2 & w_2 & 1 \\ z3 & w_3 & 1 \end{vmatrix}.$$

The constructed $\mathfrak{H}$ has determinant

$$ad - bc = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)(w_1 - w_2)(w_1 - w_3)(w_2 - w_3),$$

which is nonzero if and only if each z_i and each w_i is distinct from the rest. The case of one of the points being infinity requires division of all four determinants by this variable and then taking the limit as the variable approaches ∞, as we have done before. □

6. Geometry Comedy

It would be a crime had I not mentioned the environment where Möbius transformations naturally act. $\text{Möb}(\mathbb{C})$ is the transformation group of the half-plane model of the hyperbolic plane $\mathbb{H}^2$; here, we define a transformation group G on space X as a nonempty set of bijections of X supplied with the composition operation and satisfying

(1) G is closed under composition;
(2) G is closed under inverses.

In *Geometries* and in my paper on geodesic coding, it is established that $\text{Möb}(\mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^2)$. This is done by first defining a metric on the hyperbolic plane, seeing that it may be expressed as a function of the cross-ratio of points and their intersections with the absolute along a geodesic passing through them, and then noting that the invariance of cross-ratios by Möbius transformations implies preservations of distance. As *Geometries* shows, Möbius transformations are vital in describing the isomorphisms between the three discussed models for the hyperbolic plane. For example, converting between the half-plane and Poincaré disk requires the use of the map

$$\Omega : \mathbb{D}^2 \to \mathbb{H}^2, \qquad \Omega : z \mapsto i \cdot \frac{1+z}{1-z}.$$

As an aside, notice that, geometrically, a Möbius transformation may be obtained by first performing stereographic projection from the plane to the unit 2-sphere, S^2, rotating and moving the sphere to a new location and orientation in space, and then performing stereographic projection (from the new position of the sphere) to the plane.
There is just too much for me to say on this subject, so I omit it :p

7. Relatively Speedy Application

In the Cayley-Klein model for the hyperbolic plane, hyperbolic straight lines correspond to open chords of the disk $\mathbb{D}^2$. Let's take some chord and parameterize it by parameter x within $(-1, 1)$. And let $v \in \mathbb{R}$ be such that $|v| < c \in \mathbb{R}^\times$ fixed. Consider the map

$$T_v : [-1, 1] \to [-1, 1], \qquad x \mapsto \frac{x + v/c}{xv/c + 1}.$$

We could easily show that T_v is bijective on $[-1, 1]$ to itself, leaving endpoints in place and being an isometry on $(-1, 1)$ with respect to the hyperbolic distance:

$$d(A, B) := \frac{1}{2} \left| \log \frac{|AX|}{|BX|} : \frac{|AY|}{|BY|} \right|.$$

Thus, this isometry is, in a sense, a parallel shift along the given hyperbolic line by the vector v/c. The composition of two parallel shifts by vectors v/c and u/c is

$$\begin{aligned} x \mapsto \frac{x + v/c}{xv/c + 1} &\mapsto \left(\frac{x + u/c}{xu/c + 1} + v/c \right) \Big/ \left(\frac{x + u/c}{xu/c + 1} + 1 \right) \\ &= \left(x + c^2 \frac{v + u}{c^2 + vu} / c \right) / \left(c^2 x \frac{v + u}{c^2 + vu} / c + 1 \right); \end{aligned}$$

this shows the composition $T_u \circ T_v$ is exactly a parallel shift T_w, where w is defined by the formula

$$w := \frac{v + u}{1 + vu/c^2}.$$

This is exactly the formula for addition of velocities in a relativistic framework, under Lorentzian transformations.

References

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