BINARY UNIT, TRANSLATIONAL COMBINATORICS JANUARY

RAYMOND FRIEND

Abstract. There are many variations to the classic beaded-necklace problem, involving some difficult combinatorial problems. I aim to develop a means of studying a specific necklace problem: binary unit, translational.

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1. Introduction

In the allegory of a jeweled necklace, imagine placing n beads that are of two colors: ruby or sapphire. Once having finished your decorating, you tie the ends of the necklace in such a way that beads many pass over the seam, allowing you to rotate your beads around the necklace freely, still keeping them in the same rotational order. Your necklaces are such a sensation that you decide to sell them through a local vendor. The vendor requests to know how many types of necklaces you can send to him. You wonder: how many unique necklaces can you create using n beads?

Escaping this metaphor, we first consider the number of ways to uniquely tile a $1\times n$ board with only two states: 0 or 1, on or off. If we only compare tilings of such boards component-wise, there are obviously 2^n possible arrangements. However, we can establish an equivalence between tilings such that two tilings are equivalent if and only if one may be shifted enough times to achieve the other, where a *shift* right would involve translating each tile one square to the right and moving the rightmost tile to the leftmost position, and vice versa for a left shift. For example, the tilings A and B in Fig. 1 are equivalent because shifting the B to the left twice yields A. This equivalence restricts the number of unique tilings, and it will be our goal to analyze the number of unique tilings for any n.

2. Formalization and Failure

Definition 2.1 (n-Tiling)**.** Tiling T is a $1\times n$ grid of n tiles with states $T_i \in \{0,1\}$ for all $i : [0,n)$.

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(A) Tiling A (B) Tiling B

FIGURE 1. Two equivalent 5-tilings.

Definition 2.2 (Relation $\sim_n$)**.** n-Tilings A and B satisfy $A \sim_n B$ if and only if $\exists t : [0,n)$ such that $A_s = B_{s+t \pmod n}$ for all $s : [0,n)$.

Proposition 2.3. *Relation $\sim_n$ is an equivalence relation for all $n \geq 0$, i.e.*

(i) $A \sim_n A$;
(ii) $A \sim_n B \Rightarrow B \sim_n A$;
(iii) $A \sim_n B$ *and* $B \sim_n C \Rightarrow A \sim_n C$

for all n-tilings A, B, C.

Proof. We check each condition:

(i) Choose $t = 0$ to get the obvious statement $A_s = A_{s+0}$ for all $s \in [0,n)$.
(ii) If $t = t_1$ in the relation $A \sim_n B$, choose $t = n - t_1$ in the relation $B \sim_n A$.
(iii) If $t = t_1$ in the relation $A \sim_n B$, and if $t = t_2$ in the relation $B \sim_n C$, choose $t = t_3 = t_1 + t_2$ in the relation $A \sim_n C$. Then for any $s \in [0,n)$,

$$A_s = B_{s+t_1 (\mathrm{mod}\ n)} = C_{(s+t_2)+t_1 (\mathrm{mod}\ n)} = C_{s+t_3 (\mathrm{mod}\ n)}.$$

□

We may now discuss tilings on a higher level: equivalence classes. We will define some notations for sets of certain equivalence classes.

Definition 2.4. $C_n = \{[A] \mid A \text{ is an } n\text{-tiling}\}$.

Definition 2.5. For $k : [0,n)$, $C_{n,k} = \{[A] \mid \sum_{i=0}^{n} A_i = k\} \subset C_n$.

Example 2.6.

$$C_4 = \{[0000], [0001], [0011], [0101], [0111], [1111]\}.$$

$$C_{4,2} = \{[0011], [0101]\}.$$

Lemma 2.7.

$$\sum_{\mathcal{C} \in C_{n,k}} |\mathcal{C}| = \binom{n}{k}.$$

Proof. For fixed n and $k : [0,n)$, consider the set of all possible combinations of k tiles from the n. Denote this set P. Obviously, $|P| = \binom{n}{k}$. I aim to show $P = \bigcup_{\mathcal{C} \in C_{n,k}} \mathcal{C}$. Note if $p \in P$, then $\exists n$-tiling $T \in C_{n,k}$ such that $p \in T \subset \bigcup_{\mathcal{C} \in C_{n,k}} C$ by construction. Conversely, if $T \in \bigcup_{\mathcal{C} \in C_{n,k}} \mathcal{C}$, then T represents a choice of k tiles (those with value 1) form n. Thus, $T \in P$. □

Remark 2.8. First note that $\bigcup_{k=0}^{n} C_{n,k} = C_n$, and then notice the results of Lemma 2.7 are consistent with the fact that

$$\sum_{k=0}^{n} \sum_{\mathcal{C} \in C_{n,k}} |\mathcal{C}| = 2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

For sake of convenience, let's have a better way to write the inner sum.

Definition 2.9. Let the sum of the cardinalities of each congruence class within a given $C_{n,k}$ be given by

$$\mathcal{S}_{n,k} \equiv \sum_{\mathcal{C} \in C_{n,k}} |\mathcal{C}| = \binom{n}{k}.$$

One sees the Pascal structure to $\mathcal{S}_{n,k}$, satisfying the recurrence.

$$\mathcal{S}_{n,k} = \mathcal{S}_{n-1,k-1} + \mathcal{S}_{n-1,k}.$$

It is tempting to work towards an analysis of the cardinalities of each C_n by comparing the cardinalities of each $\mathcal{C} \in C_{n,k}$ to binomial coefficients, which are best represented in Pascal's Triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

FIGURE 2. Binomial coefficients $\binom{n}{k}$

1

1 1

1 2 1

1 3 3 1

1 4 2,2 4 1

1 5 5,5 5,5 5 1

1 6 6,6,3 6,6,6,2 6,6,3 6 1

1 7 7,7,7 7,7,7,7,7 7,7,7,7,7 7,7,7 7 1

FIGURE 3. Cardinalities of every $\mathcal{C} \in C_{n,k}$

In comparing these two tables, we notice a pattern, extrapolate, and generate a proposition:

Proposition 2.10. *By the Euclidean Algorithm, we can write* $\binom{n}{k} = n \cdot q_k + r_k$, *where* $r_k : (0, n]$ *and* $q_k \in \mathbb{Z}$. *We claim that each* $\mathcal{S}_{n,k}$ *is of the form:* $\mathcal{S}_{n,k} = n + n + \ldots + n + r_k$, *where each addend represents the cardinality of a corresponding equivalence class within* $C_{n,k}$. *Thus,* $|C_{n,k}| = q_k + 1$. *Then we claim:*

$$|C_n| = \sum_{k=0}^{n} |C_{n,k}| = n + 1 + \sum_{k=0}^{n} q_k.$$

We have $q_k = \frac{1}{n}\left(\binom{n}{k} - r_k\right)$ *for* $n \neq 0$, *where*

$$r_k = \begin{cases} n & n \mid \binom{n}{k} \\ \binom{n}{k} \pmod{n} & n \nmid \binom{n}{k} \end{cases}.$$

Thus, we can rewrite the above formula as

$$|C_n| = n + 1 + \frac{2^n}{n} - \frac{1}{n}\sum_{k=0}^{n} r_k.$$

Funnily enough, I believed this proposition for about a day and a half, and even after writing some programs to automate the assignment into equivalence classes and counting, I had not realized the error in this thinking. This idea relies on the fact that the first few integers are not very symmetric when laid out as a $1 \times n$ grid. However, when I compared the output of the last formula for $|C_n|$ to my program's output (which matches the OEIS000031 list), I found a *few* discrepancies at $n = 8, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, \ldots.$ Past $n = 10$, it seems my formula only works for prime values of n. Every non-prime $n > 9$ has too much symmetry to satisfy this proposition; the values are very close for $|C_n|$, but only become worse as more symmetries arise with increasing n. Prime n are completely described by this equation, and the expression for r_k for prime n is trivial: since $\binom{p}{k} = p \cdot m$ for some $m \in \mathbb{Z}$ when $k \notin \{0, p\}$, p must divide $\binom{p}{k}$, meaning

$$r_k = \begin{cases} 1 & k \in \{0, p\} \\ p & k \notin \{0, p\} \end{cases}.$$

Thus,

$$|C_p| = p + 1 + \frac{2^p}{p} - \frac{2 + p(p-1)}{p} = 2 + \frac{2^p - 2}{p}.$$

I found the smallest counter example to this proposition: $n = 8, k = 4$. We would expect equivalence classes each of cardinality 8, since $\binom{8}{4} = 70 = 8 \cdot 8 + 6$. But consider $T = 01010101$. The equivalence class is $[T] = \{01010101, 10101010\}$. Similarly, for $S = 00110011$, the equivalence class is $[S] = \{00110011,\ 01100110,\ 10011001,\ 11001100\}$. Thus, our integer partition is $\binom{8}{4} = 8 \cdot 8 + 4 + 2$. (The reason why 8 is the first power of 2 that causes a symmetry issue is because a 1×8 grid offers three levels of equal dissection, rather than two from a 1×4 grid, and one from a 1×2 grid).

Perhaps there is a more general formula, involving nested modular arithmetic or rounding rules. This type of complexity is very related to integer partitions, and I'm no *Ramanujan*. However, we can still easily prove why each element of $C_{n,k}$ has a cardinality less than n.

Theorem 2.11. *For all $n \geq 0$, $k : [0, n]$, and $\mathcal{C} \in C_{n,k}$, $|\mathcal{C}| \leq n$.*

Proof. Taken any $A \in \mathcal{C}$. We claim A can only satisfy $A \sim_n B$ for up to n such B. This follows by definition: $|\mathcal{C}| = |\{B \mid B \sim_n A\}|$, but $A \sim_n B$ is equivalent to having the existence of some $t : [0, n)$ such that $A_s = B_{s+t \pmod n}$ for all $s : [0, n)$. There are only n components to A and B and n possible values of t, so the number of unique B is only n. Thus, $|\mathcal{C}| \leq n$. $\square$

3. Queasy Solution

Lemma 3.1 (Burnside, Cauchy-Frobenius)**.** *Let G be a finite group that acts on a set X. For each $g \in G$, let X^g denote the set of elements in X that are fixed by g, i.e. $X^g = \{x \in X \mid gx = x\}$. We assert the following formula for the number of* *orbits:*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g| \,.$$

Proof. We re-express the sum over the group elements $g \in G$ as an equivalent sum over the set elements $x \in X$:

$$\sum_{g \in G} |X^g| = |\{(g, x) \in G \times X \mid gx = x\}| = \sum_{x \in X} |G_x| \,,$$

where $G_x = \{g \in G \mid gx = x\}$ is the stabilizer subgroup of G that fixes point $x \in X$. The orbit-stabilizer theorem says there is a natural bijection for each $x \in X$ between the orbit of $x, Gx = \{gx \mid g \in G\} \subseteq X$, and the set of left cosets G/G_x of its stabilizer subgroup G_x. With Lagrange's theorem, this implies

$$|Gx| = [G : G_x] = |G| \,/\, |G_x| \,.$$

Our sum over the set X may therefore be rewritten as

$$\sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G_x|}{|Gx|} = |G_x| \sum_{x \in X} \frac{1}{|Gx|}.$$

Finally, notice that X is the disjoint union of all its orbits in X/G, which means the sum over X may be broken up into separate sums over each individual orbit.

$$\sum_{x \in X} \frac{1}{|Gx|} = \sum_{A \in X/G} \sum_{x \in A} \frac{1}{A} = \sum_{A \in X/G} 1 = |X/G| \,.$$

Putting everything together gives the desired result:

$$\sum_{g \in G} |X^g| = |G| \cdot |X/G| \,.$$

□

Now I am going to present a non-implicit, but hardly explicit, solution to this problem, including for the case of c possible colors for each tile. With more colors, we can enumerate them $0, 1, ..., c-1$. But first, let's stick to 2 colors. Let's consider the rotation group G of the necklace: it is a cyclic group of order n. Let α be a generator of G, meaning α is a rotation of order n, such as the rotation by one bead in the positive direction. Therefore, $G = \{\alpha^1, \alpha^2, \ldots, \alpha^n\}$.
For $k : [1, n]$, the rotation α^k is a permutation of order $f_k = \frac{n}{(k,n)}$, where (k, n) denotes the greatest common factor of k and n. This is simply due to the fact that we require a number f_k such that $k \cdot f_k \equiv 0 \pmod{n}$, and f_k is the smallest such non-negative integer satisfying this. Therefore, the rotation α^k partitions the set of n beads into (k, n) non-overlapping orbits, each of size $\frac{n}{(k,n)}$.
We say a coloring is *invariant* under α^k if and only if it is constant on each orbit; that is the new notion of equivalence. With 2 colors, the number of invariant colorings for α^k is $2^{(k,n)}$, since there are (k, n) distinct orbits. According to Burnside's lemma, the number of indistinguishable colorings is obtained by averaging the number of invariant colorings over all elements of the group:

$$S_n = \frac{1}{n} \sum_{k=1}^{n} 2^{(k,n)}.$$

We can show the identity that

$$\sum_{k=1}^{n} 2^{(k,n)} = \sum_{d|n} \phi(d) 2^{n/d},$$

where $\phi(d) = \left|\left\{k : [1, n] \mid (k, n) = \frac{n}{d}\right\}\right|$, or simply the number of positive integers up to n that have a greatest common factor with n equal to $\frac{n}{d}$. This definition is equivalent to

$$\phi(n/d) = \left|\{k : [1, n] \mid (k, n) = d\}\right|.$$

We generalize to many colors by replacing the 2 in every formula with a c, obtaining:

$$S_n^{(c)} = \frac{1}{n} \sum_{d|n} \phi(d) c^{n/d}.$$

This solution involves knowing how to calculate the sum of $\phi(d)$ over all divisors of n, a heavily studied subject.