Full Achievement in Groups

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Overview

Achievement Set

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- **Strong** $\langle \overline{S} \rangle$: the set of all products of elements of S *in any order that is finitely rearranged from the original order of* S, with each element appearing at most once as a factor.

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Convention: The empty product equals the identity element of G.

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Let $G = S_4$, *and* $S = ((12), (24))$. *Then*

$$\begin{aligned} \langle S \rangle &= \{id, (12), (24), (24)(12)\} \\ &= \{id, (12), (24), (142)\} . \end{aligned}$$

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$$\begin{aligned} \langle \overline{S} \rangle &= \{id, (12), (24), (24)(12), (12)(24)\} \\ &= \{id, (12), (24), (142), (124)\}. \end{aligned}$$

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- *In* $G = S_4$, *the sequence* $S = ((12), (24))$ *is both* **weakly** *and* **strongly** *achieving in* G, *since each element is uniquely achieved by a product in* S.
- *In* $G = \mathbb{Z}_4$, *the sequence* $S = (1, 2)$ *is* **weakly** *achieving since*

$$\langle S \rangle = \{0, 1, 2, 3\},$$

but is not **strongly** *achieving since* $1 + 2 = 3 = 2 + 1$.

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Definition (Alternate)

A finite sequence S is

- **weakly achieving** iff $|\langle S\rangle| = 2^{|S|}$,
- **strongly achieving** iff $\left|\langle \overline{S}\rangle\right| = \lfloor |S|! \cdot e\rfloor$.

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- If S is weakly [strongly] achieving in G with order k, then any induced subsequence of S is weakly [strongly] achieving in G as well. So any order from 0 to $|S|$ is weakly [strongly] achieving.
- Any non-Abelian group G contains $a \neq b$ such that $ab \neq ba$, so G is strongly achieved for orders $k = 0, 1, 2$.

Finitely Generated Abelian Groups

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Theorem

Every finitely generated Abelian group is isomorphic to a unique group of the form

$$\mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_m}$$

for some non-negative integers r, m, *and* $2 \leq n_1 \mid \cdots \mid n_m$.
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Remark

An Abelian group is finite iff it is of rank $r = 0$.

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- A direct product of cyclic groups $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_m}$ is weakly achieved for any $k \leq \sum_{i=1}^{m} \lfloor \lg n_i \rfloor$ by
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- Notice a suitable order for $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_m}$ is less than or equal to
$$\left\lfloor \lg \prod_{i=1}^{m} n_i \right\rfloor = \left\lfloor \sum_{i=1}^{m} \lg n_i \right\rfloor \geq \sum_{i=1}^{m} \lfloor \lg n_i \rfloor.$$

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- $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ *has maximum weakly suitable* $k = \lfloor \lg(3 \times 6) \rfloor = 4 > 3 = 1 + 2$, *and has* 16 *weakly achieving sequences of order* 4, *including*

$$S = ((0,1),(1,1),(1,3),(1,5)) .$$

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- $\mathbb{Z}$ is weakly achieved for any $k \in \mathbb{N}$ since $\mathbb{Z}_{2^k}$ is weakly achieved in this order, and $\mathbb{Z}_{2^k}$ may be embedded in $\mathbb{Z}$.

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- $\mathbb{Z}^r$ is weakly achieved since $\mathbb{Z}$ may be embedded in $\mathbb{Z}^r$ for any $r > 0$.
- For any $r > 0$, $\mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_m}$ is weakly achieved since $\mathbb{Z}^r$ may be embedded within it.

Finitely Generated Abelian Groups (cont.)

Theorem

Any finitely generated Abelian group with rank $r > 0$ or with

$$\left\lfloor \lg \prod_{i=1}^{m} n_i \right\rfloor = \sum_{i=1}^{m} \lfloor \lg n_i \rfloor$$

is weakly achieved for any suitable order.

Direct Product on Achievement

Theorem

Suppose $G = A_1 \times \cdots \times A_m$, with each component being a group having maximum **weakly** *achieved order k_i. Then G is* **weakly** *achieved in order $k \leq \sum_{i=1}^{m} k_i$.*

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Semi-Direct Product on Strong Achievement

Semi-direct product does not respect strong achievement.
There are 4 groups represented by $C_8 \rtimes C_2$, two of which are

- $\mathbf{C_8} \times \mathbf{C_2}$ is Abelian, so is only strongly achieved by order $k \leq 1 < 2 = 1 + 1$.
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In general, $C_n \rtimes C_2$ could include $C_n \times C_2$ or D_n. For $n \geq 8$, D_n is only strongly achieved by order $k \leq 2$.

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  No: $S = ((12), (23), (124))$, since
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- Does S being weakly achieving imply each permutation of S is too?
  No: $S = ((12), (13), (234))$, since
  $$(13)(234) = (1342) = (234)(12).$$

Achievement Sets in S_n

Strongly achieving sequences are still relatively abundant in S_n:

Example

There are (at least) n strong achievement sequences in S_n of order $k = n - 1$ up to rearrangement, of the form:

$$S = ((12), (13), \dots, (1n)) .$$

Result about Strong Achievement

Definition

If S is a subset of group G, denote by $K_G(S)$ the *Orientizer* of S in G, defined by

$$K_G(S) := \{g \in G \mid \exists s, t \in S : gs = tg\}.$$

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Theorem

Suppose S is a strongly achieving subset of group G with $|S| \geq 2$. Then $S \cap K_G(S) = \varnothing$.

Result about Strong Achievement (cont.)

Remark

Note that for any subset S of G,

$$Z(G) \subseteq C_G(S) \subseteq N_G(S) \subseteq K_G(S),$$

and so if S is strongly achieving in G with order $k \geq 2$,

$$S \subseteq G \setminus K_G(S) \subseteq G \setminus N_G(S) \subseteq G \setminus C_G(S) \subseteq G \setminus Z(G).$$

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Remark

We have no such result for weakly achieving $S \subset G$, since even $S \cap Z(G)$ may be nonempty, such as how for any Abelian G, $Z(G) = G$.

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Theorem (R. Jones, 2011)

If $(x_n) \subset \mathbb{R}$, and if for each positive integer k we have

$$|x_k| > \sum_{n=k+1}^{\infty} |x_n|,$$

then $\langle (x_n) \rangle$ is a central Cantor set.

- $\left\langle \left(\frac{2}{3^n}\right)_{n=1}^{\infty} \right\rangle$ is the middle third Cantor set with endpoints 0 and 2, and is weakly achieved.

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- $\left\langle \left(\frac{1}{2^n} - \frac{1}{3^n}\right)_{n=0}^{\infty} \right\rangle = \left[0, \frac{1}{2}\right]$ is not weakly achieved.
- $\left\langle \left(\frac{1}{2^n} + \frac{1}{3^n}\right)_{n=0}^{\infty} \right\rangle$ is a central Cantor set with endpoints 0 and $\frac{7}{2}$, and is weakly achieved.

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- $\left\langle \left(\frac{1}{2^n} + \frac{1}{3^n}\right)_{n=0}^{\infty} \right\rangle$ is a central Cantor set with endpoints 0 and $\frac{7}{2}$, and is weakly achieved.
- $\left\langle \left(\frac{1}{2^n} + \frac{1}{(-3)^n}\right)_{n=0}^{\infty} \right\rangle = \left[0, \frac{3}{4}\right] \cup \left[2, \frac{11}{4}\right]$ is not weakly achieved, since $\frac{11}{24}$ is expressible as the sum of two distinct subsequences of S.

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A possible variation: Given natural number n and set G, what is the *minimum* order of a weak achievement set using n elements from G?

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- *If* $G = \mathbb{Z}$*, call this minimum number* $g(n)$*. We have for* $k \in \mathbb{N}$*:*

$$g(2k) \leq k^2 + k + 1, \qquad g(2k+1) \leq k^2 + 2k + 2.$$

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- *If* $G = \mathbb{Z}^+$, *call this minimum number* $g^+(n)$. *We have for* $k \in \mathbb{N}$:

$$g^+(n) \leq \frac{n^2 + n + 2}{2}.$$

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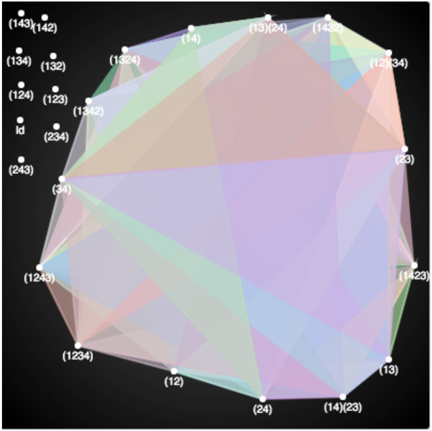
- Are there nontrivial conditions that, along with weakness, imply strong achievement?
- Can we enumerate the weakly or strongly achieving sequences in S_n?

Curious Discovery

In $G = S_4$, with strong order $k = 3$, no 3-cycle was involved in a strong achievement sequence. Why?



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- *Strongly Achieving in $\mathbb{Z}_4$:*

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- However, I do not foresee a counterexample existing in groups of order less than 25.
- We can easily see that all groups of prime order have unique sets of weakly achieving sequences.
- We can attempt to reconstruct a group and its Cayley Table from its set of weakly achieving sequences.

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- In general, for two groups of order larger than 4, they may only have the "same" set of weakly achieving sequences if there exists a bijection between them that *preserves inverses* and *respects the identity*. Stronger conditions can be applied for even larger groups.

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$$\therefore \mathscr{S}(\mathbb{Z}_8) \not\approx \mathscr{S}(Q_8).$$

Sources

- R. Jones, *Achievement Sets of Sequences*, MAA Monthly **118** (June-July 2011), 508–521.

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