AN EXPANSION ON SERIES
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Abstract. This paper resembles more an initial jab at, or a tangential reverberation off of, than an assiduous excavation into series, but we prove a result on what we coin *total sums* of power series, and see some consequences. This is another old problem of mine.

Contents

1. Introduction

High school and undergraduate students are bombarded with problems of taking derivatives and integrals of many functions, learning some advanced techniques to solving some of math's most trivial problems. Differential equations take these routines to another level of application and procedure. A natural question when considering differential equations is to wonder how many derivatives of your function f you may sum; and why stop at derivatives? Why not sum over antiderivatives of f as well? When I thought of this problem, I was looking outside my bathroom through the small window that I was barely able to stand tall enough to see through. The symbols $f, f', f'', \int f(x)dx$, and so on scrambled across the pane. I stopped the water running in the tub and went to my room to explore.
Admittedly, the work was tedious at first, and I had many mental mishaps and ruined notes from perpetuated errors. Eventually, I found a similarity between the forms of derivatives and antiderivatives of f, and this is what we will explore. I chose the name *Droz-Wolf* as the name of the theorem because it was my freshman-year multivariable calculus professor: Prof. Daniel Droz who entertained this idea with delight and excitement, pushing me to further it and consider some generalizations; and a majority of my final discoveries happened during my first night as an undergraduate student in Wolf Hall at Pennsylvania State University. My temporary roommate chose not to ask what I was doing that night.

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2. Taylor's Theorem

The first theorem is terribly unmotivated, arising from pure abstraction. If I had more free time, I would spend longer on motivating and explaining the steps leading up to this theorem.

Theorem 2.1. *Let f, g be functions defined on a closed interval $[a,b]$ that admit finite n-th derivatives on (a,b) and continuous $(n-1)$-th derivatives on $[a,b]$. Suppose $c \in [a,b]$. Then for each $x \in [a,b], x \neq c, \exists \xi$ in the segment joining c and x such that*

$$\left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k\right) g^{(n)}(\xi) = f^{(n)}(\xi) \left(g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!}(x-c)^k\right).$$

Proof. For simplicity, first assume $a < c < x < b$. Keep x fixed and consider

$$F(t) = f(t) + \sum_{k=1}^{n-1} \frac{f^{(k)}(t)}{k!}(x-t)^k,$$

$$G(t) = g(t) + \sum_{k=1}^{n-1} \frac{g^{(k)}(t)}{k!}(x-t)^k,$$

for each $t \in [c,x]$. Then F, G are continuous on $[c,x]$ and admit finite derivate on (c,x). By the mean value theorem, we may write

$$F'(\xi)[G(x) - G(c)] = G'(\xi)[F(x) - F(c)]$$

for some choice of $\xi \in (c,x)$. This gives that

$$F'(\xi)[g(x) - G(c)] = G'(\xi)[f(x) - F(c)]$$

since $F(x) = f(x)$ and $G(x) = g(x)$. But we see that by cancelling terms with opposite signs, that

$$F'(t) = \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t),$$

$$G'(t) = \frac{(x-t)^{n-1}}{(n-1)!} g^{(n)}(t),$$

which gives the desired formula when choosing $t = \xi$. The proof works equally well for the case $a < x < c < b$, choosing $\xi \in (x,c)$. □

Corollary 2.2 (Taylor's Theorem)**.** *We get Taylor's theorem with $g(x) = (x-c)^n$, namely, for some ξ, we have*

$$\left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k\right) n! = f^{(n)}(\xi)(x-c)^n,$$

or

$$f(x) = \frac{f^{(n)}(\xi)}{n!}(x-c)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Note that $g^{(k)}(c) = 0$ if $k = 0, 1, 2, ..., n-1$, and $g^{(n)} = n!$.

Taylor's theorem requires of f to be a function defined on a closed interval $[a,b] \subset \mathbb{R}$ that admits finite n-th derivatives on (a,b) and continuous $(n-1)$-th derivatives on $[a,b]$ for specified n.

3. Total Sums

Take function f which is a finite power series or polynomial with coefficients $(a_n) \subset \mathbb{R}$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

As for some notation, for some $m \in \mathbb{Z}$, define $f^{(m)}(x)$ as

$$f^{(m)} = \begin{cases} \frac{d^m f}{dx^m} & m > 0 : \text{the } m\text{-th derivative,} \\ f(x) & m = 0 : \text{identity on } f, \\ \underbrace{\int \int \cdots \int}_{m} f(x) dx^{-m} & m < 0 : \text{the } m\text{-th antiderivative.} \end{cases}$$

Also, allow us to extend the domain of the standard factorial function to any integer, $n \in \mathbb{Z}$, in this way:

$$n! = \begin{cases} n! & n \geq 0, \\ 1 & n < 0. \end{cases}$$

Theorem 3.1 (Droz-Wolf)**.** *For some bi-infinite sequence $(a_p)_{-\infty}^{\infty} \subset \mathbb{R}$, we take the power series $f(x) = \sum_{p=0}^{\infty} a_p x^p$. Denote with F the sum of each $f^{(m)}(x)$, called the total sum of f.*

$$F(x) = \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{m=-\infty}^{\infty} a_{p+m}(p+m)!.$$

The total sum is interpreted as the sum of every derivative and antiderivative of the power series f.

Lemma 3.2 (Positive Derivatives)**.** *We claim for $m \geq 1$ that*

$$f^{(m)}(x) = \sum_{p=m}^{\infty} a_p x^{p-m} \frac{p!}{(p-m)!} = \sum_{p=0}^{\infty} a_{p+m} x^p \frac{(p+m)!}{p!}.$$

Proof. We are given

$$f(x) = \sum_{p=0}^{\infty} a_p x^p.$$

Then we see

$$f^{(1)}(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \sum_{p=0}^{\infty} a_p x^p = \sum_{p=1}^{\infty} a_p x^{p-1} p + \frac{d}{dx} [a_0] = \sum_{p=1}^{\infty} a_p x^p \frac{p!}{(p-1)!}.$$

The second representation of the sum follows from the simple transformation $p \mapsto p-1$. Next, we assume that for some $m = k \in \mathbb{Z} \geq 1$, the statement holds

$$f^{(k)}(x) = \sum_{p=k}^{\infty} a_p x^{p-k} \frac{p!}{(p-k)!}.$$

The other representation follows from applying the transformation $p \mapsto p-k$. Then we wish to show the formula holds for $m = k+1$.

$$\begin{aligned} f^{(k+1)}(x) = \left(f^{(k+1)}\right)^{(1)}(x) &= \frac{d}{dx}\sum_{p=k}^{\infty} a_p x^{p-k}\frac{p!}{(p-k)!} \\ &= \sum_{p=k+1}^{\infty} a_p x^{p-k-1}(p-k)\frac{p!}{(p-k)!} + \frac{d}{dx}\left[x^0 a_k \frac{k!}{0!}\right] \\ &= \sum_{p=k+1}^{\infty} a_p x^{p-(k+1)}\frac{p!}{(p-(k+1))!}. \end{aligned}$$

Similarly, the other representation of the sum comes from the simple transformation $p \mapsto p-(k+1)$. ☐

Lemma 3.3 (Negative Derivatives)**.** *We claim for $m \leq -1$ that*

$$f^{(m)}(x) = \sum_{p=m}^{\infty} a_p x^{p-m}\frac{p!}{(p-m)!} = \sum_{p=0}^{\infty} a_{p+m} x^{p}\frac{(p+m)!}{p!}.$$

Proof. We are given

$$f(x) = \sum_{p=0}^{\infty} a_p x^p.$$

Then we see

$$f^{(-1)}(x) = \int f(x)dx = \int \sum_{p=0}^{\infty} a_p x^p dx = \sum_{p=0}^{\infty} a_p x^{p+1}\frac{1}{p+1} + a_{-1} = \sum_{p=-1}^{\infty} a_p x^{p+1}\frac{p!}{(p+1)!},$$

where we chose a_{-1} from the given bi-infinite sequence to serve as the constant of integration. The second representation of the sum follows from the simple transformation $p \mapsto p+1$. Next, we assume that for some $m = k \in \mathbb{Z} \leq -1$, the statement holds

$$f^{(k)}(x) = \sum_{p=k}^{\infty} a_p x^{p-k}\frac{p!}{(p-k)!}.$$

The other representation follows from applying the transformation $p \mapsto p+k$. Then we wish to show the formula holds for $m = k-1$ (we are performing reversed induction to prove our formula over all negative integers).

$$\begin{aligned} f^{(k-1)}(x) &= \left(f^{(k)}\right)^{(-1)}(x) \\ &= \int \sum_{p=k}^{\infty} a_p x^{p-k}\frac{p!}{(p-k)!}dx \\ &= \sum_{p=k}^{\infty} a_p x^{p-k}\frac{1}{p-k+1}\frac{p!}{(p-k)!} + a_{k-1} \\ &= \sum_{p=k}^{\infty} a_p x^{p-(k-1)}\frac{p!}{(p-(k-1))!} + a_{k-1}x^{(k-1)-(k-1)}\frac{(k-1)!}{((k-1)-(k-1))!} \\ &= \sum_{p=k-1}^{\infty} a_p x^{p-(k-1)}\frac{p!}{(p-(k-1))!}. \end{aligned}$$

Similarly, the other representation of the sum comes from the simple transformation $p \mapsto p + (k-1)$. □

We now see the ultimate result: for any $m \in \mathbb{Z}$:

$$f^{(m)}(x) = \sum_{p=0}^{\infty} a_{p+m} x^p \frac{(p+m)!}{p!}.$$

Proof. Droz-Wolf. By definition and using the two lemma just proved,

$$\begin{aligned} F(x) = \sum_{m=-\infty}^{\infty} f^{(m)} &= \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{p+m} x^p \frac{(p+m)!}{p!} \\ &= \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{m=-\infty}^{\infty} a_{p+m}(p+m)!. \end{aligned}$$

□

We can use the Taylor expansion to any f satisfying the conditions of Taylor's theorem and extend its definition to a bi-infinite sequence $(a_p)_{-\infty}^{\infty}$, and then apply this theorem to calculate the total sum. Surely the bi-infinite sequence (a_p) has its restrictions for convergence, and we assume the equality signs during divergent cases truly mean mutual divergence. As a special case, a polynomial has the special property that $\exists$ minimal $N \in \mathbb{N} \cup \{0\}$ such that $n \geq N$ implies $a_n = 0$, and $N \leq M$ for any M satisfying the same condition. Moreover, we can restrict our consideration to having for all $n < 0$: $a_n = 0$. Then

$$F(x) = \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{m=-p}^{N-p} a_{p+m}(p+m)!.$$

We could perform the transformation $m \mapsto m + p$ and get

$$F(x) = \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{m=0}^{N} a_m m! = e^x \sum_{m=0}^{N} a_m m!.$$

If we define $A = \sum_{m=0}^{N} a_m m!$ then we have finally that $F(x) = Ae^x$. This is a pretty surprising but wonderful result!

4. Totaling Total Sums

Let's denote the total sum operator from the space of power series to itself, with $\mathscr{F} : \mathscr{P}_\infty \to \mathscr{P}_\infty$, transforming a power series f as follows:

$$\mathscr{F}(f) := \sum_{m=-\infty}^{\infty} f^{(m)} = \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{m=-\infty}^{\infty} a_{p+m}(p+m)!.$$

Naturally, we could apply this operator more than once.

Proposition 4.1. *The formula for the repeated total sum operator is*

$$\mathscr{F}^n(f) = \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} a_{p+m_1+\ldots+m_n}(p + m_1 + \ldots + m_n)!.$$

Proof. The $n = 1$ case is clear from Theorem 3.1. Now assume for $n = k$ that

$$\mathscr{F}^k(f) = \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{(m_1,...,m_k)\in\mathbb{Z}^k} a_{p+m_1+...+m_k}(p + m_1 + ... + m_k)!.$$

We wish to show the corresponding statement for $n = k + 1$, that

$$\mathscr{F}^{k+1}(f) = \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{(m_1,...,m_{k+1})\in\mathbb{Z}^{k+1}} a_{p+m_1+...+m_{k+1}}(p + m_1 + ... + m_{k+1})!.$$

Notice

$$\mathscr{F}^{k+1}(f) = \mathscr{F}\left(\mathscr{F}^k(f)\right) = \mathscr{F}\left[\sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{(m_1,...,m_k)\in\mathbb{Z}^k} a_{p+m_1+...+m_k}(p + m_1 + ... + m_k)!\right].$$

Label

$$\alpha_p = \frac{1}{p!} \sum_{(m_1,...,m_k)\in\mathbb{Z}^k} a_{p+m_1+...+m_k}(p + m_1 + ... + m_k)!.$$

So

$$\mathscr{F}^{k+1}(f) = \mathscr{F}\left[\sum_{p=0}^{\infty} x^p \alpha_p\right] = \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{m_{k+1}=-\infty}^{\infty} \alpha_{p+m_{k+1}}(p + m_{k+1})!$$

We have that

$$\begin{aligned}
&\alpha_{p+m_{k+1}}(p + m_{k+1})! \\
&= \frac{(p + m_{k+1})!}{(p + m_{k+1})!} \sum_{(m_1,...,m_k)\in\mathbb{Z}^k} a_{(p+m_{k+1})+m_1+...+m_k}((p + m_{k+1}) + m_1 + ... + m_k)!
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathscr{F}^{k+1}(f) &= \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{m_{k+1}=-\infty}^{\infty} \sum_{(m_1,...,m_k)\in\mathbb{Z}^k} a_{p+m_1+...+m_{k+1}}(p + m_1 + ... + m_{k+1})! \\
&= \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{(m_1,...,m_{k+1})\in\mathbb{Z}^{k+1}} a_{p+m_1+...+m_{k+1}}(p + m_1 + ... + m_{k+1})!.
\end{aligned}$$

□

Could we total these powers of total sums of power series? Let's tweak the notation to identify $\mathscr{F} \leftrightarrow \mathscr{F}_1$. Surely, we could define $\mathscr{F}_2 : \mathscr{P}_\infty \to \mathscr{P}_\infty$, called the total power total sum operator (this will be the last one named explicitly!), that acts as

$$\mathscr{F}_2(f) = \sum_{n=0}^{\infty} \mathscr{F}_1^n(f).$$

Continuing the analogy, $\mathscr{F}_1$ is defined as

$$\mathscr{F}_1(f) = \sum_{n=-\infty}^{\infty} \mathscr{F}_0^n(f),$$

where $\mathscr{F}_0^n = f^{(n)}$. However, notice in our expansion of $\mathscr{F}_2$,

$$\mathscr{F}_2(f) = \sum_{n=0}^{\infty} \mathscr{F}_1^n(f) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} a_{p+m_1+\ldots+m_n}(p+m_1+\ldots+m_n)!$$

$$= \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{n=0}^{\infty} \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} a_{p+m_1+\ldots+m_n}(p+m_1+\ldots+m_n)!,$$

there are an infinite number of ways to express the same $z \in \mathbb{Z}$ as the index for the term $a_{p+m_1+\ldots+m_n}$ as we work over all possible $\mathbb{Z}^n$. Thus, any nontrivial f would have $\mathscr{F}_2(f)$ be divergent. Otherwise, $\mathscr{F}_2(f) = 0$ trivially. And this even occurs for all powers of $\mathscr{F}_1$ greater than or equal to 2. One can easily see that for any $z \in \mathbb{Z}$, we could express $z = m_1 + m_2$ for $m_1, m_2 \in \mathbb{Z}$. This system has infinite solutions in $\mathbb{Z}^2$, so the sum $\mathscr{F}_1^{n\geq 2}(f)$ is divergent. This is the end of our expedition, unfortunately.