# The Algorithmic Theory of Information and its Applications to Geometric Measure Theory

Doctoral Dissertation Defense


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# Background

# Hausdorff Measures

We work over a metric space $(\Omega, d)$ and consider a typical family of outer measures on this space.

**Definition ($\mathcal{H}^s$-Measures)**

For any $s \geq 0$, define the $s$**-dimensional Hausdorff outer measure** of a subset $X \subseteq \Omega$ to be:

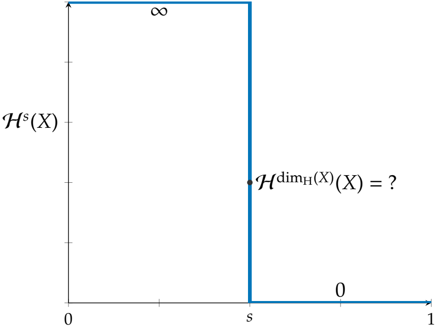
$$\mathcal{H}^s(X) = \sup_{\delta > 0} \inf_{(U_i)_{i \in \omega}} \left\{ \sum_i \operatorname{diam}(U_i)^s : \operatorname{diam} U_i \leq \delta \text{ and } \bigcup_i U_i \supseteq X \right\}.$$

**Observation**:

- $\mathcal{H}^s(X) < \infty$ implies $\mathcal{H}^{s_+}(X) = 0$ for all $s_+ > s$.
- $\mathcal{H}^s(X) > 0$ implies $\mathcal{H}^{s_-}(X) = \infty$ for all $0 \leq s_- < s$.

# Hausdorff Dimension

The map $s \mapsto \mathcal{H}^s(X)$ looks like a step function:




This motivates the definition for the **Hausdorff dimension** of a set $X$ :

$$\dim_H(X) := \inf\{s > 0 : \mathcal{H}^s(X) = 0\}.$$

# Mathematical Context

A primary goal of *Geometric Measure Theory* (GMT) is to understand the fractal properties of sets (e.g., measure and dimension).

**Examples**:

- **Marstrand-Mattila Projection Theorem**: on achieving large Hausdorff dimension under orthogonal projections,
- **Kakeya Problem**: on the Hausdorff dimension of sets which largely intersect with a line in each direction,
- **Besicovitch's Theorem**: on the existence of closed subsets of positive, finite $\mathcal{H}^s$-measure.

## Mathematical Context

A primary goal of *Algorithmic Information Theory* (AIT) is to study a robust notion for the information content of a piece of data and for algorithmic randomness.

**Three Paradigms**:

- **Unpredictability**: algorithmic mass spreading or betting strategies,
- **Typicality**: algorithmic randomness tests,
- **Incompressibility**: complexity w.r.t. universal Turing machines.

## Mathematical Context

Progress in the 2000s in connecting the GMT and AIT pursuits:

- **Correspondence Principle**: a pointwise formula for Hausdorff dimension for simple sets,
- **Point-to-Set Principle**: this formula relativized applies to general sets,
- **Finer point-to-set principles**: pointwise formulas and conditions for the Hausdorff $\mathcal{H}^s$-measures.

**Theorem (Point-to-Set Principle for $\dim_{\mathrm{H}}$ over $\mathbb{R}^n$; J. Lutz & N. Lutz 2018)**

*For any $X \subseteq \mathbb{R}^n$, we have*

$$\dim_{\mathrm{H}} X = \min_{B \in 2^{\leq \omega}} \sup_{x \in X} \dim^B(x).$$

# Applying AIT to GMT

- The power of PTS is in proving *lower bounds* for $\dim_{\mathrm{H}}$: if $B$ is an oracle witnessing PTS for $X$, then

$$(\exists x \in X)[\dim^B(x) \geq s] \implies \dim_{\mathrm{H}} X \geq s.$$

- The effective dimension of a real $x$ is the limiting value of *normalized Kolmogorov complexity*: $K(x \upharpoonright r)/r$, along its dyadic approximations. So, we may argue on the level of dyadic approximations:

$$(\forall^\infty r)[K(x \upharpoonright r) \geq^+ sr] \implies \dim(x) \geq s.$$

- Thus, AIT might offer a finer language for deducing geometric measure theoretic results: they follow from statements about individual points and their approximations.

# Where My Dissertation Fits In

1. **How does effective dimension behave under conditioning?**
   - Establish stronger robustness properties of effective dimension over Euclidean space.
2. **How does effective dimension distribute along function graphs?**
   - Elucidate the distribution of effective dimension over Euclidean space.
3. **Can we develop AIT for GMT on a broader class of metric spaces?**
   - Leverage the combinatorial structure of *nets* to algorithmically characterize fractal properties over a more generic class of metric spaces.
4. **Put these ideas to work.**
   - Witness GMT results via effective arguments.

# Robustness

# Computability Theory

Let $M$ be a Turing machine (TM) from natural numbers to natural numbers.

- $M$ **computes** a partial function $\Phi_M :\subseteq \omega \to \omega$ satisfying

  $$[n \in \mathrm{dom}(\Phi_M) \iff M(n)\downarrow] \quad \text{and} \quad [n \in \mathrm{dom}(\Phi_M) \implies M(n)\downarrow = \Phi_M(n)].$$

- $M$ **computably enumerates** $A \subseteq \omega$ if $A$ is the range of $\Phi_M$.
- There exists a **universal** TM **U** which simulates all other TMs.
- One may relativize these notions for oracle machines which also accept oracles of the form $B \in 2^\omega$.
- **Lower-semicomputability**: the ability to computably approximate a quantity or function from below, i.e.,
  - For $x \in \mathbb{R}$: if its left-Dedekind cut $\{q \in \mathbb{Q} : q < x\}$ is c.e.,
  - For $f : \omega \to \mathbb{R}$: if its lower graph $\{(\sigma, q) \in \omega \times \mathbb{Q} : q < f(\sigma)\}$ is c.e., or often expressible as having a computable function lower-approximating $f$.

# Incompressibility & Kolmogorov Complexity

The **Kolmogorov complexity** of some data $x$ captures the *minimal length of a program which produces $x$*. Fixing some universal Turing machine $\mathbf{U}$ (able to simulate all other Turing machines on all inputs), define:

$$C(x) = \min\left\{\mathrm{len}(\mathrm{program}) : \mathbf{U}(\mathrm{program}) = x\right\}.$$

There is also a conditional version:

$$C(x \mid y) = \min\left\{\mathrm{len}(\mathrm{program}) : \mathbf{U}(\langle \mathrm{program}, y\rangle) = x\right\}.$$

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More often, a universal *prefix-free* Turing machine $\mathbf{U}_{\mathrm{PF}}$ is used, which is only able to halt on a prefix-free set of programs. This defines **prefix complexity**:

$$K(x) = \min\left\{\mathrm{len}(\mathrm{program}) : \mathbf{U}_{\mathrm{PF}}(\mathrm{program}) = x\right\}.$$

$$K(x \mid y) = \min\left\{\mathrm{len}(\mathrm{program}) : \mathbf{U}_{\mathrm{PF}}(\langle \mathrm{program}, y\rangle) = x\right\}.$$

**Theorem (Chain Rule for Prefix Complexity, Gács 1974)**

*For any finitary data $x$ and $y$,*

$$K(x, y) = K(y) + K(x \mid y) \pm [O(\log \mathrm{len}(x)) + O(1)].$$

# Basic Observations for Prefix Complexity

Because $\mathbf{U}_{\mathrm{PF}}$ can simulate any algorithm, the complexity of any input cannot be exceeded by the complexity of its output.

**Theorem**

*Given a Turing machine $\Phi$ (i.e., algorithm) and finitary input $x$, it holds that*

$$K(\Phi(x)) \leq K(x) + O_\Phi(1).$$

A corresponding version exists for conditional complexity.

**Theorem (Kraft Inequality)**

$$\sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)} \leq 1.$$

**Theorem (Chaitin's Counting Theorem 1976)**

*There exists a constant $c > 0$ such that for any $n, r \in \omega$,*

1. $\max\{K(\sigma) : \sigma \in 2^n\} = n + K(n) \pm c$, *and*
2. $|\{\sigma \in 2^n : K(\sigma) \leq n + K(n) - r\}| \leq 2^{n-r+c}$.

# Lifting $K$ to Euclidean Space

Take any subsets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$.

Following A. Shen and N. Vereshchagin, define the **conditional prefix complexity of** $X$ **given** $Y$ as:

$$K(X \mid Y) := \max_q \left\{ \min_p \{K(p \mid q) : p \in X \cap \mathbb{Q}^m\} : q \in Y \cap \mathbb{Q}^n \right\},$$

while the **prefix complexity of** $X$ is just:

$$K(X) := \min\{K(p) : p \in X \cap \mathbb{Q}^m\}.$$

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Take any $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$.

The **conditional prefix complexity of** $x$ **to precision-level** $r$ **given** $y$ **to precision-level** $s$ is:

$$K_{r|s}(x \mid y) := K\left(B_{2^{-r}}(x) \mid B_{2^{-s}}(y)\right),$$

while the **prefix complexity of** $x$ **to precision-level** $r$ is:

$$K_r(x) := K(B_{2^{-r}}(x)).$$

## Known Approximations and the Chain Rule

Restricted versions of the Chain Rule and approximations to $K_{r|s}$ are known and have been used.

Fix $m, n \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r \geq s \in \omega$.

**Lemma (Approximations by Dyadic Truncations; N. Lutz & D. Stull 2020)**

$$K_{r|s}(x \mid y) = K(x \upharpoonright r \mid y \upharpoonright s) \pm [O(\log r) + O(\log s) + O_{m,n}(1)].$$

**Theorem (Approximate Chain Rule for $K_r$; N. Lutz & D. Stull 2020)**

$$K_r(x, y) = K_r(y) + K_{r|r}(x \mid y) + [O(\log r) + O_{m,n,y}(1)].$$

# Robustness of the Lift

The definition is robust up to a reordering of optimizers:

**Proposition (Robustness of the Lift)**

$$\min_p \max_q K(p \mid q) \approx \max_q \min_p K(p \mid q).$$

So, the optimizations may be performed independently:

**Lemma (Conditional Approximation by $K$-Minimizers)**

*If $p^*, q^*, w^*$ denote the $K$-minimizers of $B_{2^{-r}}(x)$, $B_{2^{-s}}(y)$, and $B_{2^{-t}}(z)$, respectively,*

$$K_{r,s|t}(x, y \mid z) \approx K(p^*, q^* \mid w^*).$$

Both these equalities hold up to sub-linear terms:

$$\pm\left[O(\log r) + O(\log s) + O_{m,n,x,y}(1)\right].$$

# Main Result for Prefix Complexity

**Theorem (Conditional Chain Rule for Conditional Complexity)**

*Let $m, n, \ell \in \omega$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^\ell$, and $r, s, t \in \omega$. Then,*

$$K_{r,s|t}(x, y \mid z) \approx K_{s|t}(y \mid z) + K_{r|s,t}(x \mid y, z),$$

*with equality holding up to* $\pm \left[ O(\log r) + O(\log s) + O_{m,n,x,y}(1) \right]$.

# Effective Dimension

In 2002, E. Mayordomo established what is now the most commonly used definition for effective Hausdorff dimension using a quantity first studied by L. Staiger in 1989.

**Definition**

Take the **effective (Hausdorff) dimension** of a real $x \in \mathbb{R}$ to be its *lower incompressibility ratio* with respect to prefix complexity:

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r} \in [0, 1].$$

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In fact, the limit superior gives an effective analog of *packing dimension*.

**Definition**

Take the **effective packing dimension** of a real $x \in \mathbb{R}$ to be its *upper incompressibility ratio* with respect to prefix complexity:

$$\mathrm{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r} \in [0, 1].$$

# Variant 1: Mutual Dimension

One variant of effective dimension is **mutual dimension** as defined by A. Case and J. Lutz in 2014.

## Definition (Mutual Information)

Let $a, b \in 2^{<\omega}$, $x \in X \subseteq \mathbb{R}^m$, $y \in Y \subseteq \mathbb{R}^n$, and $r, s \in \omega$.

- $I(a : b) := K(a) - K(a \mid b) \approx K(a) + K(b) - K(a, b)$,
- $I(X : Y) := \min \{I(p : q) : p \in X \cap \mathbb{Q}^m \text{ and } q \in Y \cap \mathbb{Q}^n\}$,
- $I_{r:s}(x : y) := I(B_{2^{-r}}(x) : B_{2^{-s}}(y))$.

## Definition (Lower and Upper Mutual Dimensions)

$$\mathrm{mdim}(x : y) := \liminf_{r \to \infty} \frac{I_{r:r}(x : y)}{r}, \quad \text{and} \quad \mathrm{Mdim}(x : y) := \limsup_{r \to \infty} \frac{I_{r:r}(x : y)}{r}.$$

# Variant 2: Conditional Dimension

A conditional version of effective dimension was first defined by J. Lutz and N. Lutz in 2018.

**Definition (Lower and Upper Conditional Dimensions)**

$$\dim(x \mid y) = \liminf_{r\to\infty} \frac{K_{r|r}(x \mid y)}{r}, \quad \text{and} \quad \mathrm{Dim}(x \mid y) = \limsup_{r\to\infty} \frac{K_{r|r}(x \mid y)}{r}.$$

**Theorem (Chain Rule for Conditional Dimension; J. Lutz & N. Lutz 2018)**

$$\dim(x) + \dim(y \mid x) \leq \dim(x,y) \leq \dim(x) + \mathrm{Dim}(y \mid x) \leq \mathrm{Dim}(x,y) \leq \mathrm{Dim}(x) + \mathrm{Dim}(y \mid x).$$

# Effective Dimension Variants and Transformations

Using the conditional Chain Rule for $K_{r|s}$, we extend the work of J. Reimann, A. Case, and J. Lutz, confirming that conditional dimension behaves predictably under computable, uniformly continuous maps.

**Theorem (Invariance)**

*Suppose $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are both bi-computable and bi-Lipschitz continuous. Then,*

$$\dim(f(x) \mid g(y)) = \dim(x \mid y).$$

As $\dim(\cdot \mid \cdot)$ and $\mathrm{mdim}(\cdot : \cdot)$ are both invariant under these maps, we might view the effectivization of fractal geometry modulo the group of bi-computable, bi-Lipschitz continuous transformations.

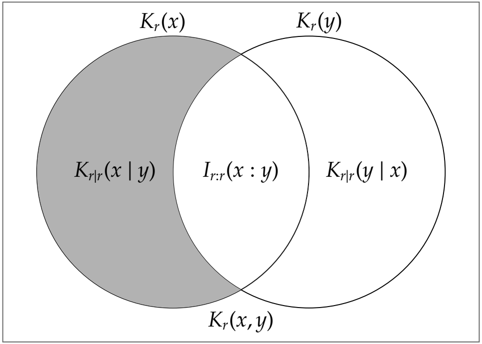
# Relating the Effective Dimension Variants

Another consequence of the conditional Chain Rule:

**Proposition (Inclusion-Exclusion, partially due to A. Case & J. Lutz 2014)**

*For each $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$,*

$$\dim(x \mid y) \leq \dim(x) - \mathrm{mdim}(x : y) \;\boldsymbol{\leq}\; \mathrm{Dim}(x \mid y),$$
$$\dim(x \mid y) \;\boldsymbol{\leq}\; \mathrm{Dim}(x) - \mathrm{Mdim}(x : y) \leq \mathrm{Dim}(x \mid y).$$



# Effective Dimension along Function Graphs

# Points on Function Graphs

**Problem**: Given $f : \mathbb{R} \to \mathbb{R}$, how does $\dim(x, f(x))$ relate to $\dim(x)$ and $f$?

- $f$ being computable and Lipschitz continuous $\implies \dim(x, f(x)) = \dim(x)$.
- Otherwise, suppose $f$ is computable given some parameters $\boldsymbol{\alpha}$. What uniform continuity properties guarantee a relation between $\dim(x, f(x))$ and $\dim(x)$, $\dim(\boldsymbol{\alpha})$, etc.?

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In 2017, N. Lutz and D. Stull established a result on the effective dimension of points on a planar line:

**Theorem (Points on Planar Lines, N. Lutz & D. Stull 2017)**

*For every $a, b, x \in \mathbb{R}$,*

$$\dim(x, ax + b) \geq \dim(x|a, b) + \min\left\{\dim(a, b), \dim^{a,b}(x)\right\}.$$

# Framework for Function Families

**Planar Lines**. Recast the family of non-vertical lines written in slope-intercept form,

$$\Phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}; \qquad \Phi(a, b, x) = ax + b.$$

Going forward, fix some slope-intercept pairs $(a, b)$ and $(u, v)$, and inputs $x_1, x_2 \in \mathbb{R}$.

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More generally, a **computable absolutely Lipschitz family (CALF)** takes the form of a partial computable map, $\Phi :\subseteq \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n$.

Fix two parameters $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$. It might happen that only certain components cause the value of $\Phi^{\boldsymbol{\alpha}} - \Phi^{\boldsymbol{\beta}}$ to vary independently of $\|x\|$, i.e.,

$$\left\|(\Phi^{\boldsymbol{\alpha}'}(x) - \Phi^{\boldsymbol{\beta}'}(x)) - (\Phi^{\boldsymbol{\alpha}}(x) - \Phi^{\boldsymbol{\beta}}(x))\right\| \leq O\left(\|\boldsymbol{\alpha}' - \boldsymbol{\alpha}\| + \left\|\boldsymbol{\beta}' - \boldsymbol{\beta}\right\|\right).$$

Collect into $(\boldsymbol{\alpha} - \boldsymbol{\beta})_{\text{cL}}$ all the rest.

# Required Property: Scaling Lipschitz Continuous

**Planar Lines**. By the triangle inequality,

$$|(a \cdot x_1 + b) - (u \cdot x_2 + v)| \leq (|x_1| + 1 + |x_1 - x_2|) \cdot \|(a, b) - (u, v)\| + |a| \cdot |x_1 - x_2|\,.$$

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**Scaling Lipschitz Continuous**. For all $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $x_1, x_2$,

$$\left\|\Phi^{\boldsymbol{\alpha}}(x_1) - \Phi^{\boldsymbol{\beta}}(x_2)\right\| \leq (O(\|x_1\| + 1) + o(1)) \cdot \left\|\boldsymbol{\alpha} - \boldsymbol{\beta}\right\| + O(\|\boldsymbol{\alpha}\| + 1) \cdot \|x_1 - x_2\|\,.$$

# Required Property: Scaling co-Lipschitz Continuous Differences

**Planar Lines**. Two lines have constant difference iff their slopes match, and

$$|a-u|\cdot|x_1-x_2| = \Big|\,[(a\cdot x_1+b)-(u\cdot x_1+v)]-[(a\cdot x_2+b)-(u\cdot x_2+v)]\,\Big|.$$

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**Scaling co-Lipschitz Continuous Differences**. For all finitary $\boldsymbol{\alpha},\boldsymbol{\beta}$, it holds that $\Phi^{\boldsymbol{\alpha}}-\Phi^{\boldsymbol{\beta}}$ is either constant or *scaling co-Lipschitz continuous*,

$$\Big\|(\boldsymbol{\alpha}-\boldsymbol{\beta})_{\mathrm{cL}}\Big\|\cdot\|x_1-x_2\| = O\left(\Big\|\Big[\Phi^{\boldsymbol{\alpha}}(x_1)-\Phi^{\boldsymbol{\beta}}(x_1)\Big]-\Big[\Phi^{\boldsymbol{\alpha}}(x_2)-\Phi^{\boldsymbol{\beta}}(x_2)\Big]\Big\|\right),$$

and there is an algorithm deciding this from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

# Required Property: Dense Intersections

**Planar Lines**. If $a$ and $b$ are $r$-dyadic, we have by Lutz & Stull,

- **Geometric Machine Construction**: may use $x \upharpoonright r$, $(ax+b) \upharpoonright r$, $r$, and $(a,b) \upharpoonright 1$ to produce a nearby line $(u_0, v_0) \in B_{2^{-r}}(a,b)$ satisfying,

$$\Big| [u_0 \cdot (x \upharpoonright r) + v_0] - (ax+b) \upharpoonright r \Big| < 2^{-r} \cdot (|u_0| + |x \upharpoonright r| + 3).$$

- **Density Argument**: there exists $r$-dyadic $(u,v) \in B_{2^{\gamma-r}}(u_0, v_0)$ such that $u \neq a$, where $\gamma = \log(2\,|a| + |x| + 5)$. Moreover, $(u,v)$ will agree in output with $(a,b)$ at input $x$ up to precision $2^{-r+\gamma}$.

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**Dense Intersections**. For all $r \in \omega$, $\boldsymbol{\alpha} \in \mathbb{D}_r^m$, and $x \in \mathbb{R}^\ell$, there is an algorithm using $x \upharpoonright r$ and $\Phi^{\boldsymbol{\alpha}}(x) \upharpoonright r$ (as well as $r$, $m$, $n$, $\ell$, and $\boldsymbol{\alpha} \upharpoonright O(1)$) to compute a $(2^{-r} \cdot O_{\|\boldsymbol{\alpha}\|,\|x\|,m,n}(1))$-approximation to some $\boldsymbol{\beta} \in B_{2^{-r} \cdot O_{\|\boldsymbol{\alpha}\|,\|x\|,m,n}(1)}(\boldsymbol{\alpha}) \cap \mathbb{D}_r^m$ satisfying $\Phi^{\boldsymbol{\alpha}} - \Phi^{\boldsymbol{\beta}}$ is scaling co-Lipschitz continuous with

$$-\log_2 \left\| (\boldsymbol{\alpha} - \boldsymbol{\beta})_{\mathrm{cL}} \right\| \leq r - O_{\|\boldsymbol{\alpha}\|,\|x\|,m,n}(1).$$

# Results for CALFs

**Theorem (Finitary Theorem)**

*Let $\Phi :\subseteq \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n$ be a CALF. Let $d < \delta \in [0, \ell]$ and $\boldsymbol{\alpha}$ and $x$ be $r$-dyadic.*

$$\textit{If for each } k \leq r: \quad K(\boldsymbol{\alpha} \upharpoonright k) \geq dk - o(k), \quad \textit{and} \quad K(x \upharpoonright k \mid \boldsymbol{\alpha}) \geq \delta k - o(k),$$

$$\textit{Then}: \quad K(x, \Phi^{\alpha}(x)) \geq K(\boldsymbol{\alpha}, x) - m \cdot \frac{K(\boldsymbol{\alpha}) - dr}{\delta - d} - o(r).$$

**Theorem (Infinitary Theorem)**

*Let $\Phi : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n$ be a CALF. Then, for every $\boldsymbol{\alpha}$ and $x$,*

$$\dim(x, \Phi^{\alpha}(x)) \geq \dim(x|\boldsymbol{\alpha}) + \min\left\{\dim(\boldsymbol{\alpha}), \dim^{\alpha}(x)\right\}.$$

**Theorem (Dimension Spectrum)**

*For every $\boldsymbol{\alpha} \in \Omega$ and $\delta \in (0, \ell)$, we have*

$$\dim_{\mathrm{H}}\left(\left\{x \in \mathbb{R}^\ell : \dim(x, \Phi^{\alpha}(x)) < \delta + \frac{\dim(\boldsymbol{\alpha})}{2}\right\}\right) \leq \delta.$$

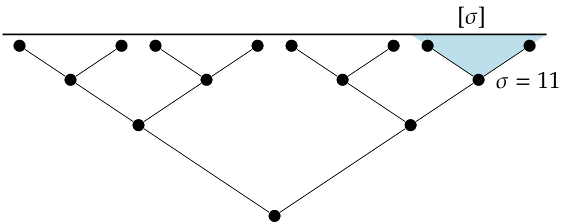
# GMT by AIT on Nets

# Cantor Space

**Cantor space**: $2^\omega$, the space of all infinite binary sequences. Viewed as the space of extensions to all the finite binary strings $2^{<\omega}$.

**Cylinder sets**: $\mathcal{B}$, the collection of all clopen cylinders:

$$[\sigma] := \{x \in 2^\omega : \sigma \preceq x\}, \quad \text{where} \quad \sigma \in 2^{<\omega}.$$



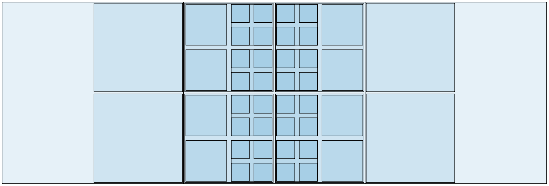

**Compatible Metric**: with respect to the product topology on $2^\omega$,

$$d(x,y) := \begin{cases} 2^{-N} & \text{if } x \neq y \text{ and } N = \min\{n \in \omega : x(n) \neq y(n)\}, \\ 0 & \text{if } x = y. \end{cases}$$

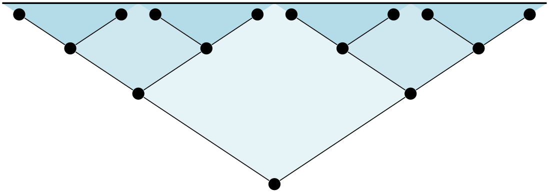
# Prototypical Nets

Nets are modeled on the prototypical countable, nested, and computable collections of subsets. These collections also facilitate AIT.

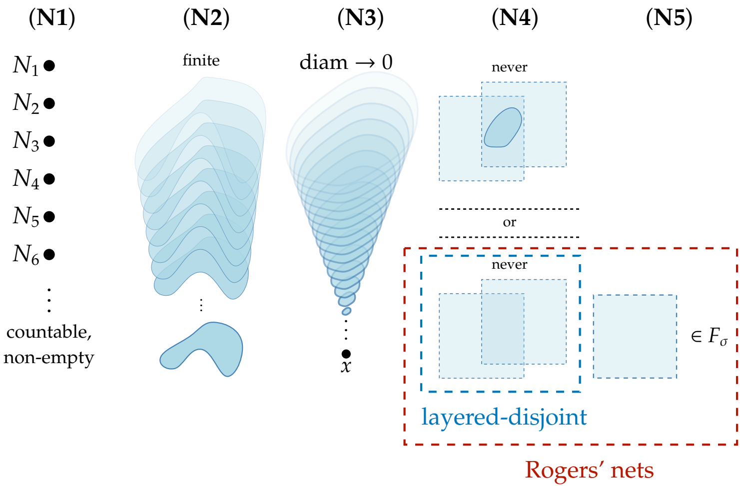
- The family $\mathcal{Q}^n$ of dyadic rational cubes in $\mathbb{R}^n$:



- The basis $\mathcal{B}$ of clopen cylinders on Cantor space $2^\omega$:



# Net Axioms


(N1)
(N2)
(N3)
(N4)
(N5)
$N_1$
$N_2$
$N_3$
$N_4$
$N_5$
$N_6$
countable, non-empty
finite
diam $\to 0$
$x$
never
or
never
$\in F_\sigma$
layered-disjoint
Rogers' nets

## Basic Observations about Nets

**Observations**:

- $\subseteq$-relation on $\mathcal{N}$ induces a tree structure, and a **rank**:

$$\mathrm{rank}(N) := 1 + \max\left\{\mathrm{rank}(N') : N \subsetneq N' \in \mathcal{N}\right\}.$$

- For any $x \in \Omega$, denote the **set of representations:** $\mathcal{R}(x)$, of $x$:

$$(\forall n)[x \in N_{i_n}] \quad \text{and} \quad \mathrm{diam}(N_{i_n}) \to 0 \text{ as } n \to \infty.$$

- Net spaces generalize Mayordomo's *nicely covered spaces*.
- Any separable, ultrametric space possesses many nets. This holds for the spaces: $2^\omega$, $\mathbb{R}^n$, $\omega^\omega$, and $\mathbb{Q}_p$.

# Measures on Metric Spaces

**Definition**

An **outer measure** on metric space $(\Omega, d)$ is a function $\mu : \mathcal{P}(\Omega) \to [0, +\infty]$ satisfying

- **Monotonicity**: $X \subseteq Y \subseteq \Omega \implies \mu(X) \leq \mu(Y)$,
- **Countable Subadditivity**: $\mu\left(\bigcup_i X_i\right) \leq \sum_i \mu(X_i)$.

There is a standard method for producing outer measures from premeasures.

**Theorem (Rogers' "Method II")**

*If $\rho$ is a premeasure on $C \subseteq \mathcal{P}(\Omega)$, define for each $X \subseteq \Omega$ and $\delta > 0$:*

$$\mathcal{H}^{\rho}_{\delta}(X) := \inf_{(C_i)_{i\in\omega} \subseteq C} \left\{ \sum_i \rho(C_i) : \mathrm{diam}_d(C_i) \leq \delta, \bigcup_i C_i \supseteq X \right\}, \quad \text{and} \quad \mathcal{H}^{\rho}(X) := \sup_{\delta > 0} \mathcal{H}^{\rho}_{\delta}(X).$$

*Then $\mathcal{H}^{\rho}$ is an outer measure on $\Omega$.*

# Hausdorff Measures

**Definition**

Any non-decreasing, right-continuous map $h : [0, \infty) \to [0, \infty]$ with $h(t) = 0 \iff t = 0$ is a **dimension function**.

Associate to any dimension function $h$ a **Hausdorff premeasure**:

$$\rho_h(X) := (h \circ \mathrm{diam}_d)(X),$$

and its corresponding Method II outer measure $\mathcal{H}^h$.

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The family of $s$**-dimensional Hausdorff premeasures**:

$$\rho_s := \rho_{h_s}, \quad \text{where} \quad h_s : t \mapsto t^s \quad \text{for any} \quad s \geq 0,$$

and their corresponding Method II outer measures $\mathcal{H}^s$.

# Net Measures

**Definition**

A *net premeasure* is a function $\rho$ defined on $\mathcal{N} \cup \{\emptyset\}$ such that

$$\rho(\emptyset) = 0 \quad \text{and} \quad 0 \le \rho(N) \le +\infty \quad \text{for all} \quad N \in \mathcal{N}.$$

Fix a procedure from premeasures to outer measures *in a net* $\mathcal{N}$:

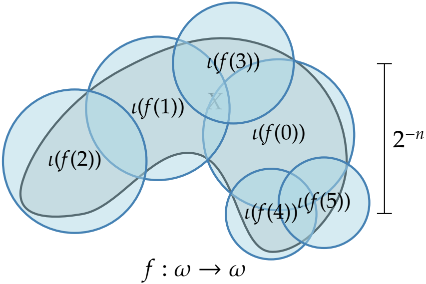
**Method II (in $\mathcal{N}$)** : premeasure $\rho \mapsto$ net measure $(\mathcal{H} \upharpoonright \mathcal{N})^\rho$,

$$\text{where} \quad (\mathcal{H} \upharpoonright \mathcal{N})^\rho(X) := \sup_{\delta>0} \inf_{(N_i)\subseteq\mathcal{N}} \left\{ \sum_i \rho(N_i) : (N_i)_i \text{ a } \delta\text{-cover of } X \right\}.$$

Restricting $\rho_s$ to $\mathcal{N}$ produces the net premeasure $\rho_s \upharpoonright \mathcal{N}$, satisfying:

$$\mathcal{H}^{\rho_s \upharpoonright \mathcal{N}} = (\mathcal{H} \upharpoonright \mathcal{N})^s.$$

# Coverings Rooted in Nets




Hausdorff coverings:

**Classically**: sequences of arbitrary subsets.
**Effectively**: unif.-c.e. sequences of $\Sigma^0_1$.

Net coverings:

**Classically**: sequences of net elements.
**Effectively**: unif.-c.e. such sequences.

Define the **Hausdorff dimension of** $X$ **restricted to** $\mathcal{N}$:

$$\dim_{\mathcal{N}}(X) := \inf\{s > 0 : (\mathcal{H} \upharpoonright \mathcal{N})^s(X) = 0\}.$$

Here, covers may only use elements from the net, $\mathcal{N}$.

# Presenting a Net

*Computability*: with respect to an $\omega$-presentation $\mathscr{R}$ of the net $\mathcal{N}$, computing:

- $\mathrm{in} : \omega^2 \to \{\top, \bot\}$: the containment relation,

$$\mathrm{in}(i, j) \iff N_i \subseteq N_j,$$

- $\mathrm{pred} : \omega^2 \to \{\top, \bot\}$: the predecessor relation,

$$\mathrm{pred}(i, j) \iff N_j \subsetneq N_i \wedge (\forall N \in \mathcal{N})[N_j \subsetneq N \subseteq N_i \to N = N_i],$$

- $\mathrm{diam} : \omega \to [0, \infty]$: the diameter function,

$$\mathrm{diam}(i) = \mathrm{diam}_d(N_i) = \sup\left\{\rho(x, y) : x, y \in N_i\right\},$$

- $\mathrm{root} : \omega \to \{\top, \bot\}$: the roots of $\mathcal{N}$,

$$\mathrm{root}(i) \iff (\forall N \in \mathcal{N})[N \supseteq N_i \implies N = N_i],$$

under some indexing $\iota : \omega \to \mathcal{N}$ which respects containment.

# Developing AIT over Net Spaces

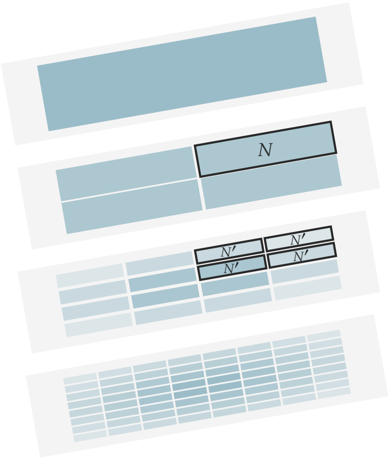
**Question**: To what extent do the robustness and PTS results of $2^\omega$ and $\mathbb{R}^n$ extend to net spaces?

**Outline**:

1. Extend complexity notions.
2. Extend algorithmic randomness notions.
3. Find sufficient conditions for the various characterizations of complexity and effective dimension to agree.
4. Conclude PTS principles.

# Net Semimeasures

**Goal**: Get complexity notions by spreading mass along nets.




Continuous Net Semimeasure:

$M : \mathcal{N} \to [0, \infty)$ such that

$$M(N) \geq \sum_{N' \text{ imm. succ. of } N} M(N'),$$

$$\text{and} \quad 1 \geq \sum_{N \text{ a root of } \mathcal{N}} M(N).$$

Discrete Net Semimeasure:

$m : \mathcal{N} \to [0, 1]$ such that

$$1 \geq \sum_{N \in \mathcal{N}} m(N).$$

# Net Semimeasures

**Idea**: a net element is *simple* when a universal prior distribution (**M** or **m**) can dedicate mass to it. Restrict semimeasures to being *lower-semicomputable*.

**Definition**

A C/D semimeasure **M** is **optimal** if for any other C/D semimeasure $M$:

$$(\exists \beta > 0)(\forall N \in \mathcal{N})[\mathbf{M}(N) \geq \beta \cdot M(N)].$$

| | | | | |
|---|---|---|---|---|
| **Continuous Semimeasures** | $\rightarrow$ | $\exists$ optimal lower-semicomputable **M** | $\rightarrow$ | *A priori* complexity: $\mathrm{KM}(\mathrm{N}) \equiv -\log \mathbf{M}(N)$ |
| **Discrete Semimeasures** | $\rightarrow$ | $\exists$ optimal lower-semicomputable **m** | $\rightarrow$ | Coding Theorem for *prefix* complexity: $\mathrm{K}(\mathrm{N}) =^{+} -\log \mathbf{m}(N)$ |

# Algorithmic Complexity for Arbitrary Points and Subsets

- We have $K$, KM for net elements $N \in \mathcal{N} \subseteq \mathcal{P}(\Omega)$.
- Equivalent, cylinder-based characterizations $K_r(x)$ on $2^\omega$:

$$G_r(x) := \inf\{K(\sigma) : [\sigma] \subseteq B_{2^{-r}}(x)\},$$
$$H_r(x) := \inf\{K(\sigma) : x \in [\sigma] \text{ and } \operatorname{diam}([\sigma]) \leq 2^{-r}\}.$$

## Definition

$(\Omega, d, \alpha)$ is a **computable metric space** if $\alpha$ is a dense sequence in $(\Omega, d)$ for which the function mapping $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is computable.

- When $(\Omega, d, \alpha)$ is a computable metric space, may define:

$$\mathbf{K}(X) := \min\{\mathbf{K}(i) : \alpha(i) \in X\}.$$

  And if $\mathcal{N}$ is also layered-disjoint and compatible with $\alpha$, we get a locally optimal outer measure on $\Omega$:

$$\boldsymbol{\kappa}(X) := 2^{-\mathbf{K}(X)}.$$

- Under stronger *grate* axioms, get $\mathbf{K}(N) \approx K(N)$ and $\mathbf{G}_r(x) \approx \mathbf{H}_r(x)$.

# Effective Dimension Notions

Fix a net space $(\Omega, \mathcal{N})$, a representation $\mathscr{R}$ of $\mathcal{N}$, and a point $x \in \Omega$.

## Definition (Unpredictability)

**Local dimension** w.r.t. a semimeasure or outer measure $\mu$:

$$\dim_{\mathrm{loc}} \mu(x) = \inf_{(i_n)_n \in \mathcal{R}(x)} \left\{ \liminf_{n \to \infty} \frac{\log \mu(N_{i_n})}{\log \operatorname{diam}(N_{i_n})} \right\}.$$

For instance, $\mathbf{M}$, $\mathbf{m}$, or $\boldsymbol{\kappa}$.

## Definition (Typicality)

**Effective Hausdorff dimension**:

$$\operatorname{effdim}(x) := \inf \{ s > 0 : x \text{ is not Martin-Löf-}\mathscr{R}\text{-}s\text{-random} \}.$$

## Definition (Incompressibility)

**Incompressibility ratio** w.r.t. some complexity notion $\mathbf{C}_r$:

$$\dim_{\mathbf{C}}(x) = \liminf_{r} \frac{\mathbf{C}_r^{\mathscr{R}}(x)}{r}.$$

# Asymptotic Coincidences

**Theorem**

*For any represented net space $(\Omega, \mathcal{N}, \mathscr{R})$ and point $x \in \Omega$,*

$$\underbrace{\dim_{\mathrm{loc}} \mathbf{M}(x) = \dim_{\mathrm{loc}} \mathbf{m}(x)}_{\textit{semimeasures}} = \underbrace{\mathrm{cdim}(x)}_{\textit{supergales}} = \underbrace{\dim_{\mathbf{G}}(x)}_{\textit{compression}} = \underbrace{\mathrm{effdim}(x)}_{\textit{effective coverings}} .$$

*If $\mathcal{N}$ is further a layered-disjoint, computable grate:*

$$\cdots = \underbrace{\dim_{\mathrm{loc}} \boldsymbol{\kappa}(x)}_{\textit{locally optimal outer measure}} = \underbrace{\dim_{\mathbf{H}}(x)}_{\textit{compression}} .$$

# Point-to-Net Principle for Net Spaces

**Theorem ("Point-to-Net Principle")**

*Given a net space* $(\Omega, \mathcal{N})$ *and* $X \subseteq \Omega$*,*

$$\dim_{\mathcal{N}}(X) = \inf_{\mathscr{R}} \dim_{\mathscr{R}}(X) = \inf_{\mathscr{R}} \sup_{x \in X} \dim_{\mathscr{R}}(x).$$

**Proof.**

Since $\dim_{\mathcal{N}}$ has no computability restriction: $\dim_{\mathcal{N}}(X) \leq \inf_{\mathscr{R}} \dim_{\mathscr{R}}(X)$.

For each $\dim_{\mathcal{N}}(X) < s \in \mathbb{Q}$, there exists a sequence $(U^s_n)_{n \in \omega}$ where each $U^s_n \subseteq \omega$ satisfies $\mathrm{DW}_s(U^s_n) \leq 2^{-n}$ and $(U^s_n)_n$ covers $X$ in the sense of an ML-type test. Take any $\omega$-presentation $\mathscr{R}$ of $\mathcal{N}$ with:

$$\mathscr{R} \geq_{\mathrm{T}} \bigoplus \{(U^s_n)_n : \dim_{\mathcal{N}}(X) < s \in \mathbb{Q}\}.$$

By definition, if $s > \dim_{\mathcal{N}}(X)$, then $X$ is covered by $(U^s_n)_n$, which is an ML-$\mathscr{R}$-$s$-test. Therefore, $s \geq \dim_{\mathscr{R}}(X)$, as well. So, by definition, $\dim_{\mathscr{R}}(X) \leq \dim_{\mathcal{N}}(X)$. □

# Point-to-Set Principle for Net Spaces

**Definition**

A premeasure $\rho$ is said to be *comparable* to $\rho_s$ if for all $X \subseteq \Omega$,

$$\mathcal{H}^{\rho}(X) = 0 \iff \mathcal{H}^{s}(X) = 0.$$

**Theorem (Point-to-Set Principle for $\dim_{\mathrm{H}}$)**

*Let $(\Omega, d)$ be a metric space. Suppose for each $s > 0$, there exists a net on $\Omega$ and a corresponding net premeasure which is comparable to $\rho_s$. Then, for each $X \subseteq \Omega$,*

$$\dim_{\mathrm{H}} X = \inf_{s>0} \inf_{\mathscr{R}_s} \sup_{x \in X} \dim_{\mathscr{R}_s}(x).$$

**Example**: Rogers & Davies showed any separable, ultrametric space exhibits this richness of net measures.

# Point-to-Net Principles for $\mathcal{H}^s$-Measures

In GMT, one often wishes to evaluate $\mathcal{H}^s(X)$, not just $\dim_{\mathrm{H}} X$. In 2024, P. Lutz & J. Miller provided pointwise formulas and conditions for these $\mathcal{H}^s$ over $2^\omega$.

We extended these to any represented net space, $(\Omega, \mathcal{N}, \mathscr{R})$.

**Theorem**

*For every $X \subseteq \Omega$ and $s \geq 0$,*

$$\log(\mathcal{H} \upharpoonright \mathcal{N})^s(X) =^+ \inf_{\mathscr{R}} \sup_{x \in X} \inf_{(i_n)_n \in \mathcal{R}(x)} \liminf_{n \to \infty} [\mathrm{KM}^{\mathscr{R}}(i_n) + s \cdot \log_2 \operatorname{diam}(i_n)],$$

*and $X$ is not $\sigma$-finite for $(\mathcal{H} \upharpoonright \mathcal{N})^s$ if and only if*

$$(\forall \mathscr{R})(\exists x \in X)[x \textit{ is strong Solovay-}\mathscr{R}\textit{-s-random}].$$

# Which Spaces have Effective Nets?

Certain metric spaces admit nets which are sufficiently effectively definable.

**Theorem**

*Effective Rogers nets may be constructed on each of:* $2^\omega$, $\mathbb{R}^n$, $\omega^\omega$, *and* $\mathbb{Q}_p$.

**Theorem**

*Suppose that* $(\Omega, d)$ *is a compact, computable metric space. Then, there exists an arithmetically definable Rogers net of* $\Sigma^0_2$*-classes over* $(\Omega, d)$.

*Moreover, if* $\mathcal{H}^s(\Omega) = 0$ *for some* $s > 0$*, then for any* $X \subseteq \Omega$*,*

$$\dim_{\mathrm{H}} X = \inf_{\mathscr{R}} \sup_{x \in X} \dim_{\mathscr{R}}(x).$$

# Application: Extending Besicovitch's Theorem

For a given metric space $(\Omega, d)$ and premeasure $\rho$ on $\Omega$, denote:

$(*) : \iff (\forall F \subseteq \Omega$ compact with $\mathcal{H}^\rho(F) > 0)(\exists E \subseteq F$ compact$)[0 < \mathcal{H}^\rho(E) < \infty]$.

**Theorem (Besicovitch 1952)**

*Let $\Omega = \mathbb{R}^n$ and $\rho = \rho_s$. Then, $(*)$ holds.*

**Theorem (Rogers & Davies 1970)**

*Let $(\Omega, \mathcal{N})$ be a net space and $\rho$ a finite net premeasure on $\mathcal{N}$ such that $\mathcal{H}^\rho$ has no infinite-measure points and such that any decreasing sequence of compact sets $(F_n)_n$ with $\mathcal{H}^\rho(\bigcup_n F_n) = 0$ has $\mathcal{H}^\rho_\delta(F_n) \to 0$ as $n \to \infty$. Then, $(*)$ holds.*

**Theorem (Joint work with E. Gruner and J. Reimann)**

*Let $\mathcal{N}$ be a scaling, finitely-branching, layered-disjoint net and a subbasis on $(\Omega, d)$ having a net Hausdorff premeasure $\rho$. Then, $(*)$ holds.*

# Incompressibility Arguments for GMT

# A Short Example

## Theorem

*Suppose there is a locally Lipschitz continuous map $f : U \times W \twoheadrightarrow X$. Then*

$$\dim_{\mathrm{H}} X \leq \dim_{\mathrm{H}} f(U \times W) \leq \dim_{\mathrm{H}}(U \times W) \leq \dim_{\mathrm{H}} U + \dim_{\mathrm{P}} W.$$

## Effective Proof.

Let $B \geq_{\mathrm{T}} f$ be a Hausdorff oracle for $X$ and $U$, and a packing oracle for $W$. For any point $x \in X$ there exists $u \in U$ and $w \in W$ such that for any $r \in \omega$,

$$K_r^B(x) = K_r^B(f(u,w)) \leq K_r^B(u,w) + o(r) \leq K_r^B(u) + K_r^B(w) + o(r).$$

Let $\varepsilon > 0$. Since $B$ is a Hausdorff oracle for $X$, there exists $x \in X$ such that $\dim_{\mathrm{H}} X \leq \dim^B(x) + \varepsilon$. Then,

$$\begin{aligned} \dim_{\mathrm{H}} X - \varepsilon \leq \dim^B(x) &= \liminf_{r\to\infty} \frac{K_r^B(x)}{r} \leq \liminf_{r\to\infty} \frac{K_r^B(u) + K_r^B(w)}{r} \\ &\leq \dim^B(u) + \mathrm{Dim}^B(w) \leq \dim_{\mathrm{H}} U + \dim_{\mathrm{P}} W. \end{aligned}$$

□

# Applications

Let $m \in \omega$ and $\|\cdot\|_*$ be a norm on $\mathbb{R}^m$.

**Theorem (Orthogonal Projections)**

*For each direction $e \in \mathbb{S}^{m-1}$ and subset $X \subseteq \mathbb{R}^m$,*

$$\dim_{\mathrm{H}}\left(\left\{\left\|\mathrm{proj}_e(x)\right\|_* : x \in X\right\}\right) \geq \dim_{\mathrm{H}} X - (m-1).$$

**Theorem (Radial Projections)**

*For each point $z \in \mathbb{R}^m$ and subset $X \subseteq \mathbb{R}^m$,*

$$\dim_{\mathrm{H}}\left(\left\{\frac{x-z}{\|x-z\|_*} : x \in X\right\}\right) \geq \dim_{\mathrm{H}} X - 1.$$

**Theorem (Pinned-Distance Sets, Altaf, Bushling & Wilson 2023)**

*For each point $z \in \mathbb{R}^m$ and subset $X \subseteq \mathbb{R}^m$,*

$$\dim_{\mathrm{H}}\left(\{\|x-z\|_* : x \in X\}\right) \geq \dim_{\mathrm{H}} X - (m-1).$$

# Orthogonal Projection Theorems

**Theorem (Marstrand-Mattila Projection Theorem 1954)**

*Let $0 < n < m$ and $X \subseteq \mathbb{R}^m$ be analytic. Then,*

$$\dim_{\mathrm{H}}(\mathrm{proj}_V(X)) = \min\{\dim_{\mathrm{H}} X, n\}, \qquad \textit{for almost all } n\textit{-dimensional subspaces } V.$$

---

**Dialogue between AIT and GMT**:

- In 2018, N. Lutz and D. Stull showed the same conclusion holds when $\dim_{\mathrm{H}} X = \dim_{\mathrm{P}} X$ and $n = 1$, via effective arguments.
- In 2021, T. Orponen extended this for all $m$ and $n$, instead via classical, combinatorial arguments.
- Important to Orponen's approach are two key lemmas. In joint work with R. Bushling supervised by J. Reimann, we prove these lemmas using incompressibility arguments.

# “$(\delta, s)$-Sets”

## Definition

Let $s > 0$, $k \in \omega$, and $C \geq 1$. A finite set $P \subseteq 2^{<\omega}$ is called a $(C, 2^{-k}, s)$**-set** if,

$$\left|\{\sigma \in P : \sigma \succeq \tau\}\right| \leq C \cdot \left(\frac{2^{-\operatorname{len}(\tau)}}{2^{-k}}\right)^{s}, \quad \text{for any} \quad \tau \in 2^{\leq k}.$$

Intuitively, “$(\delta, s)$-sets” are finite collections of points which are guaranteed to not be very concentrated when viewed with granularity larger than $\delta$.

# Decomposition Lemma

## Proposition (T. Orponen 2021)

*Let $s \leq 1$, $k \in \omega$, and $X \subseteq 2^\omega$ be covered by at most $C \cdot 2^{s \cdot k}$ many $(2^{-k})$-balls. Then, there exists a decomposition $X = X_{\text{good}} \sqcup X_{\text{bad}}$ such that:*

(i) $\mathcal{H}^s_\infty(X_{\text{bad}}) \leq 1/L$, *and*

(ii) $X_{\text{good}}$ *is contained in the* $(2^{-k})$*-neighborhood of a* $(CL, 2^{-k}, s)$*-set.*

## Proof of (i).

Define the set of "bad strings":

$$S_{\text{bad}} := \left\{ \sigma \in 2^{\leq k} : [\sigma] \cap X \neq \emptyset \quad \text{and} \quad K(\sigma) < s \cdot \text{len}(\sigma) - \log L \right\}.$$

Take $X_{\text{bad}} = [S_{\text{bad}}]$. Then, by the Kraft Inequality,

$$\mathcal{H}^s_\infty(X_{\text{bad}}) \leq \text{DW}_s(S_{\text{bad}}) = \sum_{\sigma \in S_{\text{bad}}} 2^{-s \cdot \text{len}(\sigma)} \leq \frac{1}{L} \sum_{\sigma \in S_{\text{bad}}} 2^{-K(\sigma)} \leq \frac{1}{L}.$$

$\square$

# Decomposition Lemma

**Proof of (ii).**

Put $X_{\text{good}} := X \setminus X_{\text{bad}}$, and $S_{\text{good}} := X_{\text{good}} \upharpoonright k$. Then, $X_{\text{good}}$ is contained in the $(2^{-k})$-neighborhood of $S_{\text{good}}$.

**Claim**: *$S_{\text{good}}$ is a $(CL, 2^{-k}, s)$-set.*

Fix $\tau \in 2^{\leq k}$. If $\tau \in S_{\text{bad}}$, then $[\tau] \subseteq X_{\text{bad}}$, so $S_{\text{good}}$ has no extensions of $\tau$.

Now, assume that $\tau \notin S_{\text{bad}}$. Since $X$ is covered by $C \cdot 2^{-s \cdot k}$ balls of radius $2^{-k}$, we may bound the $2^{-k}$-precision complexity of any $x \in X$:

$$K(x \upharpoonright k) \leq s \cdot k + \log C + O(1).$$

This works for any $\sigma \in X \upharpoonright k$:

$$K(\sigma) \leq s \cdot \text{len}(\sigma) + \log C + O(1).$$

# Decomposition Lemma

## Proof of (ii).

Suppose $\sigma \succeq \tau$ for $\sigma \in X \upharpoonright k$. The Coding Theorem and Counting Theorem together imply $\tau$ has at most

$$2^{K(\sigma)-K(\tau)+K(k)+O(\log \operatorname{len}(\tau))+O(1)}$$

many descriptions of length $K(\sigma) + O(\log \operatorname{len}(\tau)) + O(1)$.

But any $\sigma \succeq \tau$ generates a two-part description of this form. So, we may bound the number of extensions of $\tau$ which are "good":

$$\begin{aligned}
\left|\left\{\sigma \in X_{\text{good}} \upharpoonright k : \sigma \succeq \tau\right\}\right| &\leq \max\left\{2^{K(\sigma)-K(\tau)+K(k)+O(\log \operatorname{len}(\tau))+O(1)} : \sigma \in X \upharpoonright k\right\} \\
&\leq 2^{(s\cdot k+\log C+O(1))-(s\cdot \operatorname{len}(\tau)-\log L)+o(k)} \\
&= 2^{s(k-\operatorname{len}(\tau))+\log C+\log L+o(k)} \\
&\leq C'L\left(\frac{2^{-\operatorname{len}(\tau)}}{2^{-k}}\right)^{s}.
\end{aligned}$$

$\square$

# Open Questions

# Further Opportunities for Research

1. How does the **choice of net** (or, computable dense subset) on a metric space affect its AIT?
2. Is there a robust version of **conditional, mutual dimension**?
3. **Implications of Infinitary Theorem** for CALFs to GMT?
4. Which other GMT results admit (or, follow from) **finitary statements**?
5. Opportunities for **effectivizing the density result** over Cantor or net spaces? Reverse math, forcing notions, continuous semimeasures approach, lowness results, etc.

Thank you!

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