Moduli Spaces and $SL(2, \mathbb{R})$ Actions

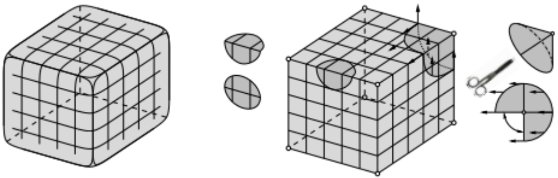
Michael Borowski   Raymond Friend   Xiao Sun

Mathematical Outing for Undergraduates, 2017

Flat Surface

The sphere has constant positive curvature, the torus has constant zero curvature, and higher genus curvature has constant negative curvature...or do they?

- Regardless of genus, we can give any surface a flat structure, but there will be singular points.
- Imagine flattening the sides and pushing all curvature to a few points.
- Look at the cube, which represents a flat sphere with eight conical singularities.

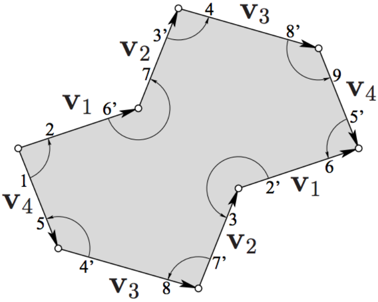


Definition

One useful definition of a translation surface, alternate to previous, is

- A closed orientable surfaces endowed with a flat metric having a finite number of conical singularities and having trivial linear holonomy.
- Trivial linear holonomy means that all cone angles at conical singularities are integer multiples of 2π.

Constructing a Translation Surface
Review



- Take $v_1, \ldots, v_n \in \mathbb{R}^2$ and create broken line.
- Then permute vectors by $\sigma \in S_n$ to create another broken line.
- Deformities preserve number of singularities and genus.

Holomorphic 1-Form

Each polygon representing the translation surface is embedded within $\mathbb{C}$.

- There exists the natural coordinate $z \in \mathbb{C}$.
- Parallel translations identify sides of the polygon

$$z' = z + \text{ct}.$$

Holomorphic 1-Form

Each polygon representing the translation surface is embedded within $\mathbb{C}$.

- There exists the natural coordinate $z \in \mathbb{C}$.
- Parallel translations identify sides of the polygon

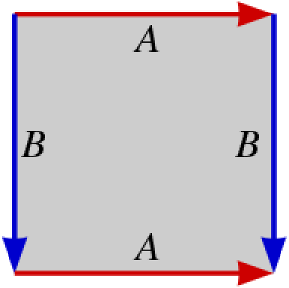
$$z' = z + \text{ct}.$$

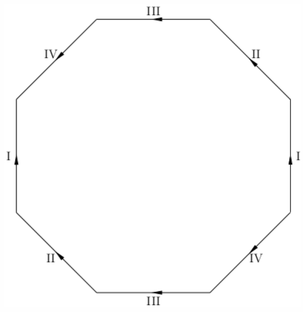
- Passing polygon to surface X, z is not globally defined anymore. But local changes are defined by translations, so in general, $dz = dz'$.
- Thus, the holomorphic 1-form dz on $\mathbb{C}$ defines a holomorphic 1-form ω on X which in local coordinates has the form $\omega = dz$, and $\omega = 0, \forall x \in \Sigma$.

Degree of Singularities

- Locally about zero, a holomorphic 1-form can be represented as $w^d dw$, where d is called the *degree* of zero.
- Conical angle about a zero of ω, a singularity, is given by $\alpha = 2\pi(d+1)$.
- Degrees of zeros determine genus

$$\sum_{j=1}^{s} d_j = 2g - 2.$$





$$\alpha_j = 2\pi(d_j + 1) \qquad \sum_{j=1}^{s} d_j = 2g - 2$$

Translating Translation Surface Terminologies

Could instead see v_j as complex number.

- If v_j joins P_j and P_{j+1}, and ρ_j is path on X joining the points, then

$$v_j = \int_{P_j}^{P_{j+1}} dz = \int_{\rho_j} \omega.$$

Classifications

Definition

Let T_g be all translation surfaces of genus g.

We will say two translation surfaces $(X, \omega) \sim (X', \omega')$ iff there is a holomorphic diffeomorphism $\phi : X \to X'$ such that $\phi_* \omega' = \omega$, and such that $\text{area}(X) = \text{area}(X')$.

Classifications

Definition

Let T_g be all translation surfaces of genus g.

We will say two translation surfaces $(X, \omega) \sim (X', \omega')$ iff there is a holomorphic diffeomorphism $\phi : X \to X'$ such that $\phi_* \omega' = \omega$, and such that $\text{area}(X) = \text{area}(X')$.

Definition

For partition of $2g - 2 : \kappa$, stratum $T_g(\kappa) \subset T_g$, where element from T_g: $(X, \omega) \in T_g(\kappa)$ iff zeros of ω match partition κ.

Assigning Coordinates from Homology

Stratum $T_g(\kappa)$ has complex dimension $2g + s - 1$, with local coordinates

$$(X, \omega) \mapsto \left(\int_{\gamma_1} \omega, ..., \int_{\gamma_n} \omega \right),$$

where $n = dim\,(H_1(X, \{x_1, ..., x_s\})) = 2g + s - 1$, and where $(\gamma_i)_{i=1}^n$ are a symplectic basis of the homology group $H_1(X, \{x_1, ..., x_s\})$.

$\mathrm{SL}(2, \mathbb{R})$ Actions

If (X, ω) was constructed from polygon P, and if $h \in \mathrm{SL}(2, \mathbb{R})$, then $h(X, \omega)$ is associated with polygon $h(P)$.

Observation

$SL(2, \mathbb{R})$ actions preserve measure and parallel lines, so we preserve strata $T_g(\kappa)$ under $h \in SL(2, \mathbb{R})$.

$\mathrm{SL}(2,\mathbb{R})$ Actions

If (X, ω) was constructed from polygon P, and if $h \in \mathrm{SL}(2,\mathbb{R})$, then $h(X, \omega)$ is associated with polygon $h(P)$.

Observation

$SL(2,\mathbb{R})$ actions preserve measure and parallel lines, so we preserve strata $T_g(\kappa)$ under $h \in SL(2,\mathbb{R})$.

- Any $h \in GL^+(2,\mathbb{R})$ action is continuous and preserves parallel lines, creating another flat surface kX.
- Subgroup $SL(2,\mathbb{R})$ is area-preserving and ergodic, meaning geodesic flow will pass over each region in X in proportion to its area.

Moduli Space

Recall that we say two translation surfaces $(X, \omega) \sim (X', \omega')$ iff there is a holomorphic diffeomorphism $\phi : X \to X'$ such that $\phi_* \omega' = \omega$, and such that $\text{area}(X) = \text{area}(X')$.

Definition

The Moduli Space $\mathcal{H}_{g,s} = \{[(X, \omega)]_\sim \mid \text{genus}(X) = g, |\Sigma| = s\}$.

Diffeomorphism Groups

Definition

Let's denote by $\text{Diff}(X, \Sigma)$ the group of homeomorphisms of M fixing each point of Σ.

- And let $\text{Diff}_0(X, \Sigma)$ the neutral component of $\text{Diff}(M, \Sigma)$.
- And let $\text{Mod}(X, \Sigma) = \text{Diff}(X, \Sigma)/\text{Diff}_0(X, \Sigma)$ the *modular group*.

Definition

Given two manifolds M, N a differentiable map $f : M \to N$ is called a *diffeomorphism* if it is a bijection and its inverse $f^{-1} : N \to M$ is differentiable as well.

Period Map

Definition

The Teichmüller space $Q(X, \Sigma, \kappa)$ is the set of orbits of the action of $\mathrm{Diff}_0(X, \Sigma)$ on the set of translation surface structures on (X, Σ, κ).

Let's denote by Θ the *period map*, taking the Teichmüller space to the cohomology group $H^1(X, \Sigma, \mathbb{C})$:

$$\Theta : Q(X, \Sigma, \kappa) \to \mathrm{Hom}(H_1(X, \Sigma, \mathbb{Z}), \mathbb{C}).$$

Proposition

The period map is a local homeomorphism.

$GL(2,\mathbb{R})$ on Teichmüller Space

- Regarding the period map Θ, the group $GL(2,\mathbb{R})$ acts on the right-hand side of $\mathrm{Hom}(H_1(X,\Sigma,\mathbb{Z}),\mathbb{C})$ by acting on the target $\mathbb{C}$.
- The period map is then covariant with respect to the actions of $GL(2,\mathbb{R})$ on the source and the image.

Torus Example

Recall the torus T: $g = s = 1$, and with a single marked point $\{A_1\}$. Fix a basis $[\zeta_1], [\zeta_2]$ for the homology group $H_1(T, \{A_1\}, \mathbb{Z})$.

$$Q(T, \{A_1\}, \mathbb{Z}) = \left\{(\zeta_1, \zeta_2) \in (\mathbb{C}^*)^2, \zeta_2/\zeta_1 \notin \mathbb{R}\right\}.$$

Moduli Space

Definition

The moduli space is the quotient

$$\mathcal{M}(X, \Sigma, \kappa) := Q(X, \Sigma, \kappa)/\mathrm{Mod}(X, \Sigma).$$

Torus Example

Considering again the torus T: $g = s = 1$.

- Then the normalized Teichmüller space is $\mathrm{SL}(2, \mathbb{R})$
- The modular group is $\mathrm{SL}(2, \mathbb{Z})$
- The moduli space is the space of normalized lattices $\mathrm{SL}(2, \mathbb{R}) / \mathrm{SL}(2, \mathbb{Z})$
- Teichmüller flow is essentially geodesic flow on the modular surface.